Sylvester rank functions, epic division rings and the strong Atiyah conjecture for locally indicable groups

Diego López Álvarez

A thesis submitted for the degree of
Doctor in Mathematics.

Supervised by:

Andrei Jaikin Zapirain

Madrid, 2021

FUNCIONES DE RANGO DE SYLVESTER, ANILLOS DE DIVISIÓN ÉPICOS Y LA CONJETURA FUERTE DE ATIYAH PARA GRUPOS LOCALMENTE INDICABLES

A lo largo de la tesis, consideramos cuestiones relacionadas con embeddings de dominios no conmutativos en anillos de división.

Por un lado, tratamos el problema de existencia de tales embeddings para anillos de grupo $K[G]$ donde K es un subcuerpo del cuerpo de números complejos $\mathbb{C}$ y G es un grupo localmente indicable (por ejemplo, un grupo libre de torsión que admite una presentación con sólo una relación). Estos grupos forman una subfamilia de grupos ordenables a izquierda, y por tanto este problema es un caso particular del problema de embedding de Malcev.
En este sentido, la conjetura fuerte de Atiyah para estos anillos de grupo, motivación original y principal de la tesis, propone un candidato a anillo de división que contiene a $K[G]$, este es, la clausura de división de $K[G]$ en el anillo clásico de cocientes $\mathcal{U}(G)$ del álgebra de grupos de von Neumann $\mathcal{N}(G)$. En el resultado principal (trabajo conjunto con A. Jaikin-Zapirain) probamos que la conjetura fuerte de Atiyah se satisface en este caso y que, además, el anillo de división resultante puede ser identificado unívocamente mediante una propiedad universal. Los resultados y métodos asociados nos permiten probar posteriormente otras conjeturas relacionadas, como por ejemplo una versión de la conjetura de aproximación de Lück para grupos virtualmente localmente indicables.

Por otro lado, consideramos la noción de universalidad de un anillo de división. Para un anillo R, un anillo de división universal de fracciones es un anillo de división que contiene a y está generado por R como anillo de división, y en el que podemos invertir "la mayor cantidad de matrices" posible sobre R. A este respecto, los dominios de Sylvester y pseudo-Sylvester son anillos que admiten un anillo de división universal de fracciones sobre el que toda matriz se vuelve invertible a menos que haya una obstrucción "obvia" que lo impida.
En un trabajo conjunto con F. Henneke probamos que los productos cruzados de la forma $E * G$, donde E es un anillo de división y G es libre-por-{cíclico infinito}, son siempre dominios pseudo-Sylvester, y exploramos la situación más general de productos cruzados $\mathfrak{F} * \mathbb{Z}$ de un anillo de ideales libres (fir) $\mathfrak{F}$ y el anillo de enteros $\mathbb{Z}$.

Durante toda la tesis, la teoría de funciones de rango de Sylvester proporciona al mismo tiempo un lenguaje homogéneo y una herramienta para abordar los problemas considerados. Por ello, analizamos en mayor profundidad el espacio de las funciones de rango de Sylvester que pueden definirse sobre ciertas familias de anillos, tales como los dominios de Dedekind o una subfamilia de anillos de polinomios de Laurent asimétricos con coeficientes en un anillo de división (trabajo conjunto con A. Jaikin-Zapirain).

A mi familia y a Rocío

Agradecimientos

Ningún camino se recorre solo, y son muchas las personas que fortuita y afortunadamente me han permitido compartir consigo parte del largo trecho que me ha llevado a escribir estas páginas, que cuentan mejor lo que he sido y lo que soy que el resto de la tesis. Y aunque quizá no es lo suyo comenzar unos agradecimientos con una disculpa, siento mucho la incapacidad para reflejar con estas pocas palabras lo que ha significado tenerlos a mi lado.

La primera persona a la que debo agradecer que esto sea posible es, sin duda alguna, Andrei. Gracias por brindarme la oportunidad de aprender contigo, y gracias por toda la ayuda a lo largo de estos poco más de 4 años, que en varias ocasiones ha trascendido a lo puramente académico. Sabes que no escribiría esto si no lo pensara, pero a tu lado he entendido lo que hace a un investigador. Muchas gracias, de corazón.

Aprovechando la oportunidad, agradezco a los profesores que me han alentado a seguir este camino. Muchísimas gracias, Vicente y Arturo, por inculcarme la fascinación por las Matemáticas en el colegio y el instituto. Muchísimas gracias, Margarita, porque si no te hubiera tenido como tutora (no oficial) no habría sabido seguir adelante tras la entrada en la universidad. Muchísimas gracias, Mercedes, por enseñarme a disfrutar el Álgebra, porque seas o no consciente de ello inspiraste a muchos con tu pasión por tu trabajo. Y muchísimas gracias también a Luis Felipe y Fernando, que durante el Máster y la Escuela JAE, respectivamente, me guiaron durante los primeros pasos por la investigación. Junto con ellos, agradezco muchísimo a Yago y a Javier A. su acogida dentro del grupo; desconocen lo que un simple gesto puede significar para alguien.

Me gustaría agradecer también a Javier Sánchez, cuya tesis ha sido ejemplo e inspiración de la que aquí se presenta, tanto por el contenido como por el mimo que se palpa al leerla. También quiero agradecerle su labor como lector. In this regard, I would like to thank Hanfeng Li, whose thorough reading and comments on the thesis have helped me a lot in improving the outcome. I would also like to thank him and Lukasz Grabowski for writing the international reports to the thesis.

Ahora, comenzando por el principio, gracias a mis amigos de toda la vida, a Raquel, a Carlos, a mis compañeros de clase en el colegio e instituto, a la gente de la plaza de Realejo Alto y a mis compañeros del Osusona™. Aunque en algunos casos nuestros caminos se hayan alejado un poco con el paso del tiempo, me resulta imposible escribir unos agradecimientos que no los incluyan. Gracias por los mejores años de mi vida, cuando nuestra mayor preocupación era saber quién traería la pelota el sábado.

Gracias también a todas aquellas personas con las que compartí gran parte de la etapa universitaria. En especial a Luis M., Misael, Yeray, Aythami y Juan Jesús, por compartir sus locuras conmigo y permitirme compartir las mías. El camino se hace más ligero cuando vas bien acompañado, y no lo habría disfrutado ni la mitad si no me hubiera topado con ustedes.

Muchísimas gracias a todos los que, entre la UAM y el ICMAT, han aliviado las cargas de este viaje, que como todo viaje largo ha tenido sus momentos críticos.

No puedo empezar de otra manera que no sea dando las gracias a los compañeros del despacho 513. Marta, durante muchos años has sido la voz de mi conciencia, así que gracias por tu insistencia, tu ánimo y tu apoyo con todo. Sinceramente, he conocido a muy pocas personas como tú en este tiempo, y haber coincidido contigo tantos años y en tantos sitios sólo me hace pensar en la suerte que me ha acompañado siempre. Carlos, ya en aquel momento se podían ver tus aptitudes de padre y docente. Gracias por haberme prestado atención cuando más aislado me sentía en la investigación, y por haberme echado un cable cuando creía no entender nada. Ale, Fran, aunque finalmente pudimos compartir poco tiempo como compañeros de despacho por causas inesperadas, agradecí muchísimo tenerlos allí; devolvieron al 513 la vida que perdía a medida que se iba quedando vacío.

Agradezco también a Adri, doctorando Prime y cafetero premium. Gracias por todas esas charlas de café a primera hora de la mañana que, durante mucho tiempo, se convirtieron en lo personal en la mayor motivación del día. A Julio, al que de alguna manera siento mi hermano de tesis y con el que he compartido muchos momentos de frustración, de charlas, de congresos y de marrones varios (gracias a los cuales sabemos preparar y servir cantidades industriales de café, té y pastas). Espero que algún día te des cuenta de lo que vales en realidad. A Luis A., al que me gusta afirmar que no soporto cuando en realidad pienso que hace falta más gente como él por el mundo. Eres de las personas que hacen grupo, que acogen y hacen sentir cómoda a la gente cuando aún está desubicada, pero negaré rotundamente que he escrito esto si me preguntan.

Y por supuesto, le doy las gracias a todas las personas que han formado parte de este camino a lo largo de los años: a Jaime, Bea B., Bea P., Raquel, Florian, Diego M., Hang Lu, Manuel, Nikita, Dani, Marcos, Irina, Iason, Glenier, Sauli, Leyter, Martí, José, a mi hermano mayor de tesis Javier, y al menor Jan, y a las incorporaciones más recientes Celia, Dani, Mikel e Ismael.

I would also like to thank all the people who tried hard to make me feel like home during my stay at the MPIM in Bonn. Wolfgang, thank you for giving me the opportunity to be there, and Benny, Johanna, Elia, Claudio, I am sincerely grateful for letting me spend this adventure with you. Fabian, I am not grateful but indebted to you. You, more than anyone, experienced by my side the tough part of the stay, and there is absolutely nothing I can write here to express how much I owe you for that. You have not only been a fantastic host and office mate during those three months and a great collaborator after them, but a friend to me. I wish you the best luck someone can wish for, because you do deserve it. Thank you.

Tampoco me puedo dejar atrás a Diego A. y Tania, que me han acogido siempre en su casa, primero en Madrid y luego en Bonn. Hemos sido compañeros de clase, de piso, de despacho, . . . y en todas las ocasiones me han tendido su mano. Muchísimas gracias. Y gracias también a Víctor, con quien he convivido en Madrid la gran mayoría de estos años, confinamiento incluido, y que es excepcional como compañero y persona.

Y he dejado para el final a las personas a las que más debo, por aguantarme desde que no tenía idea alguna hasta que se me ocurrió la “brillante” idea de aventurarme con la tesis: mi familia. Le doy las gracias a mi tío Javi, que siempre me infundió ánimos para seguir este camino, a Marian y a mis abuelos Domingo y Mercedes. A mi hermano Enri, que como hermano mayor ha tenido que abrir camino y, por qué no decirlo, sacarme las castañas del fuego en más de una ocasión antes y después de la etapa universitaria, y a su pareja Laura, a quien le ha tocado la ardua tarea de meterlo en vereda. Y por supuesto, a mis padres, Enrique y Candelaria, que son quienes me han traído hasta aquí y han puesto todos los medios y todo el apoyo para guiarme hasta lo que soy. En especial querría agradecerle a mi madre, que siempre se ha partido el lomo para darnos el empujón necesario para sacarnos adelante, y que aún cree que lo que hace por el resto de la familia no es suficiente.

A Lana y a Mimi, mis dos pequeñas y el amor más puro, incondicional e inocente que uno puede recibir. Lana, me habría gustado que pudieras seguirme hasta este momento, pero aunque no haya sido posible siempre te llevo conmigo. Mimi, gracias por devolverme la sonrisa, por ser mi compañera de juegos y por sacarme de paseo siempre que me ves. Aunque ninguna de las dos sabría leer estas páginas, me gusta pensar que entienden por qué forman parte de ellas.

Y finalmente quiero dar las gracias a Rocío. Son tantas las decisiones difíciles que hemos debido tomar a lo largo de estos últimos años, incluso en la distancia, que ya ni imagino ni entiendo una vida sin ti como compañera de viaje. Gracias por estar ahí, porque en ocasiones saber que cuento contigo es lo que me mantiene en pie. Ojalá la conclusión de este capítulo en mi vida sea el inicio de nuestra vida juntos.

En definitiva, gracias a todos por todo. Cuanto más escribo y recuerdo más consciente soy de que sin su compañía este trayecto no habría cobrado sentido. Cruzo los dedos por no haberme olvidado de nadie, y si es así, discúlpame, también va por ti. GRACIAS.

Contents

Introducción y conclusiones

Aunque quizá, si hablamos del desarrollo cronológico de la tesis, esto no sea del todo preciso, las principales motivaciones de este trabajo pueden explicarse desde la perspectiva de un primer curso en álgebra lineal y teoría de anillos.

A este respecto, una de las primeras estructuras algebraicas con la que nos topamos al iniciarnos en ambos campos es la de cuerpo (conmutativo), siendo $\mathbb{Q}$, $\mathbb{R}$ o $\mathbb{C}$ los ejemplos más habituales. El hecho de que todo elemento no nulo en un cuerpo K admite un inverso multiplicativo y de que todo K-módulo (es decir, espacio vectorial) finitamente generado es libre con un número finito fijo de elementos en cualquier base, hace de los cuerpos un contexto ideal en el que trabajar.

Como consecuencia, dado un anillo conmutativo R, es natural preguntarse si R puede ser identificado como subanillo de algún cuerpo, de manera que aún podamos utilizar la maquinaria desarrollada en álgebra lineal y heredemos de paso algunas de sus propiedades básicas. Por ejemplo, el anillo de los número enteros, $\mathbb{Z}$, es de manera natural un subanillo de $\mathbb{Q}$, $\mathbb{R}$ y $\mathbb{C}$. Así pues, podríamos plantearnos, quizá de manera aún imprecisa, las siguientes cuestiones:

1. Dado un anillo conmutativo R, ¿existe un cuerpo K del que R sea subanillo?

2. Si para un anillo R concreto la respuesta a la primera pregunta es afirmativa, ¿es el cuerpo K "único"?

3. Si, tras precisar el significado de unicidad, la respuesta a la segunda pregunta es negativa, ¿existe alguna manera de caracterizar (alguno de) los cuerpos de los que R es subanillo? ¿Existe alguno que sea "universal" en algún sentido?

Tratemos de puntualizar y responder apropiadamente estas preguntas en el caso conmutativo. Un obstáculo insalvable para la existencia de un cuerpo K como en 1. es la presencia de divisores de cero no triviales en R, es decir, de elementos no nulos a y b tales que $ab = 0$ en R. No obstante, si no estamos en esta situación, o equivalentemente, cuando R es un dominio, sabemos que podemos construir el denominado cuerpo de fracciones de R, un cuerpo cuya descripción recuerda a la de $\mathbb{Q}$ con respecto a $\mathbb{Z}$, en el sentido de que sus elementos son fracciones de elementos de R con denominador no nulo junto con las operaciones usuales. Por tanto, la respuesta a la primera pregunta es afirmativa si y sólo si R es un dominio.

Con respecto a la segunda pregunta, ya hemos mencionado un ejemplo en el que contamos con múltiples opciones entre las que elegir (de hecho, una cantidad no numerable de ellas si consideramos extensiones de cuerpos de $\mathbb{Q}$), pero algunas de ellas son o bien muy "grandes" o bien muy "complicadas" en comparación con el anillo original $\mathbb{Z}$. En este sentido, la construcción de $\mathbb{Q}$ a partir de $\mathbb{Z}$ lo hace quizá el cuerpo más "sencillo" de entre todos ellos: simplemente añadimos un inverso $\frac{1}{b}$ para todo entero no nulo b y los elementos estrictamente necesarios (fracciones $\frac{a}{b}$) para dotar de una estructura de anillo al nuevo conjunto obtenido. Decimos en este caso que $\mathbb{Q}$ está generado por $\mathbb{Z}$ como cuerpo, puesto que todo número racional puede obtenerse a partir de números enteros por medio de sumas, restas, multiplicaciones e inversiones. En vista de esta discusión, y dado que a partir de un cuerpo K podríamos obtener otros tantos por medio de extensiones, parece razonable restringir la pregunta 2. al caso de cuerpos que están generados por R (en el sentido previo), y considerar unicidad salvo isomorfismos que hagan conmutativo el siguiente diagrama

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ K \arrow[rr, "\cong"] & & K'. \end{tikzcd}$$

Observemos además que, como en el caso de $\mathbb{Z}$ y $\mathbb{Q}$, el cuerpo de fracciones $\mathcal{Q}(R)$ de R contiene el menor número de elementos necesario para construir un cuerpo a partir de R, y por tanto está contenido en cualquier otro cuerpo que contenga a R. De manera más precisa, todo homomorfismo inyectivo de R en un cuerpo K se extiende a un homomorfismo inyectivo de $\mathcal{Q}(R)$ en K. Por tanto, $\mathcal{Q}(R)$ es, salvo un isomorfismo como el arriba indicado, el único cuerpo que contiene y está generado por R.

Resumiendo, las respuestas a las preguntas anteriores son las siguientes.

1. Existe un cuerpo que contiene a R si y sólo si R es un dominio.

2. Si R es un dominio, su cuerpo de fracciones es el único cuerpo (salvo un isomorfismo como el mencionado anteriormente) que contiene y está generado por R como cuerpo.

3. La respuesta a la pregunta 2. es afirmativa.

De esta manera, y tras especificar que estamos interesados en cuerpos generados por el anillo original, todas las preguntas tienen una respuesta satisfactoria en el caso conmutativo.

En el caso no conmutativo, el concepto análogo de cuerpo es el de anillo de división (o cuerpo no conmutativo) $\mathcal{D}$. Como en el caso de un cuerpo conmutativo, todo elemento no nulo en un anillo de división tiene inverso multiplicativo y todo $\mathcal{D}$-módulo a izquierda (y derecha) finitamente generado es libre con un número finito fijo de elementos en cualquiera de sus bases. Además, salvo por aquellas que dependen de la conmutatividad del producto, gran parte de las propiedades de cuerpos son ciertas también sobre anillos de división. Por ello, y una vez analizadas las respuestas en el caso anterior, es natural adaptar y extender las preguntas previas a este nuevo contexto.

1'.- Dado un dominio no conmutativo R, ¿existe un anillo de división $\mathcal{D}$ del que R sea subanillo?

2'.- Supongamos que la respuesta a la primera pregunta es afirmativa para R, y que $\mathcal{D}$ está generado por R como anillo de división. ¿Es $\mathcal{D}$ único? Es decir, si $\mathcal{D}'$ es otro anillo de división que contiene y está generado por R, ¿existe un isomorfismo que haga el diagrama

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr, no head] & \\ \mathcal{D} \arrow[rr, "\cong"] & & \mathcal{D}'. \end{tikzcd}$$

conmutativo?

3'.- Si la respuesta a la segunda pregunta es negativa, ¿existe una manera de caracterizar (alguno de) los anillos de división que contienen a R? ¿Existe alguno que sea "universal" en algún sentido?

Desafortunadamente, en este caso las respuestas a estas preguntas no son tan satisfactorias. En cuanto a la primera, A. I. Malcev demostró en [Mal37] que existen dominios que no admiten homomorfismos inyectivos a ningún anillo de división (véase también [Coh06, Exercises 2.11, 9]), mientras que en cuanto a la segunda existen dominios R que admiten varios anillos de división no isomorfos que los contienen y están generados por éstos (véase, por ejemplo, [Fis71], o [Sán08, Corollary 7.13]).

No obstante, a pesar de las dificultades para abordar estas preguntas en el caso general, la teoría de anillos de división épicos desarrollada por P. M. Cohn en los años setenta (véase [Coh06, Chapter 7]) responde de manera abstracta a las preguntas 1'. y 2'. Esta teoría trata el problema más general de encontrar homomorfismos (no necesariamente inyectivos) de un anillo R (no necesariamente dominio) a anillos de división $\mathcal{D}$ de manera que $\mathcal{D}$ esté generado por la imagen de R (esta última propiedad es precisamente lo que significa el adjetivo "épico" en este contexto). P. M. Cohn caracterizó a estos últimos, salvo un isomorfismo como el descrito en 2', en función de las matrices sobre R cuya imagen en $\mathcal{D}$ es invertible. En este sentido, definió el anillo de división épico universal $\mathcal{U}$ para R como aquel con la propiedad de que, si una matriz A sobre R se vuelve invertible en algún anillo de división, entonces es también invertible en $\mathcal{U}$. Por supuesto, a priori no hay razón para que tal anillo de división exista en general.

Esta caracterización de anillos de división épicos en términos de matrices es también análoga a la del caso general conmutativo. Para un anillo conmutativo R, un homomorfismo $\phi : R \to K$ a un cuerpo épico K está completamente determinado por su kernel $\ker \phi$, que es siempre un ideal primo de R. De manera más precisa, dado un par (K, ϕ) como el anterior, el cuerpo K es isomorfo al cuerpo residual del anillo local $R_{\ker \phi}$, la localización de R en el ideal primo $\ker \phi$, y dos ideales primos diferentes dan lugar mediante este procedimiento a dos cuerpos épicos no isomorfos, dado que los elementos de R que se vuelven invertibles en cada uno de ellos (es decir, los que yacen fuera del ideal) son diferentes. Esto nos da una correspondencia biyectiva entre el conjunto de cuerpos épicos salvo isomorfismo y el conjunto de ideales primos del anillo.

En la misma dirección, P. M. Cohn definió la noción de ideal matricial primo y demostró que existe una correspondencia biyectiva entre R-anillos de división épicos $(\mathcal{D}, \phi)$ (salvo el isomorfismo definido en 2') y los ideales matriciales primos de R. En analogía con el caso anterior, dado un R-anillo de división épico $(\mathcal{D}, \phi)$, la colección de matrices cuadradas que van a parar a través de ϕ a matrices singulares sobre $\mathcal{D}$, denominado el núcleo singular de ϕ, constituye un ideal matricial primo $\mathcal{P}$, y $\mathcal{D}$ es (isomorfo a) el anillo de división residual del anillo local no trivial $R_{\mathcal{P}}$, la localización de R en $\mathcal{P}$ (que puede definirse a partir de una presentación de R añadiendo formalmente inversos para las matrices cuadradas que no están en $\mathcal{P}$).

Observemos que este procedimiento no genera nuevos objetos en el caso conmutativo, dado que una matriz cuadrada A sobre un anillo conmutativo R se vuelve invertible al aplicar el homomorfismo $\phi : R \to K$ si y sólo si el determinante de $\phi(A)$, que es un elemento de $\phi(R)$, es no nulo. Por tanto, $\ker\phi$ contiene toda la información sobre el núcleo singular.

P. Malcolmson reformuló en [Mal80] esta caracterización de anillos de división épicos de varias maneras. Una de ellas fue a través de funciones de rango algebraicas (conocidas hoy en día como funciones de rango de Sylvester sobre matrices), que se postularán como herramienta y lenguaje central en esta memoria. Entrando un poco más en detalles, tal y como ocurre en un cuerpo conmutativo, en un anillo de división puede definirse el rango $\mathrm{rk}_{\mathcal{D}}(A)$ de una matriz A de tamaño $n \times m$, por ejemplo, como el menor entero no negativo k tal que A admite una descomposición $A = BC$ con B de tamaño $n \times k$ y C de tamaño $k \times m$. Además, una matriz cuadrada A es invertible sobre $\mathcal{D}$ si y sólo si tiene rango máximo. De esta forma, un homomorfismo $\phi : R \to \mathcal{D}$ induce una "función de rango con valores enteros" sobre R, definida sobre una matriz sobre R como el $\mathrm{rk}_{\mathcal{D}}$-rango de su imagen por ϕ. Así, las matrices cuadradas sobre R que se vuelven invertibles son precisamente las de rango máximo, y el complemento (en el conjunto de matrices cuadradas sobre R) de este conjunto es el núcleo singular del homomorfismo.

El rango sobre un anillo de división puede ser caracterizado a partir de algunas de sus propiedades, lo que nos permite axiomatizar y formalizar la noción de función de rango de Sylvester sobre matrices en un anillo R (véase el Capítulo 1). El resultado de P. M. Cohn puede reformularse en este lenguaje como sigue:

Teorema (Cohn,Malcolmson). *Existe una correspondencia biyectiva entre la colección de R-anillos de división épicos salvo isomorfismo y la colección de funciones de rango de Sylvester sobre matrices en R que toman valores enteros.*

A lo largo de este documento, consideraremos casos particulares de las preguntas 1' y 3', y problemas que tienen por objetivo entender mejor el espacio de funciones de rango de Sylvester asociado a un anillo R.

Con respecto a la pregunta 1', un problema abierto que recibe habitualmente el nombre de "problema de embedding de Malcev" ([KM18, Problema 1.6]), plantea si la respuesta a 1' es afirmativa para anillos de grupo $K[G]$ donde K es un cuerpo conmutativo y G es un grupo ordenable a izquierda. A. I. Malcev ([Mal48]) y B. H. Neumann ([Neu49]) probaron de manera independiente que, efectivamente, este es el caso si G es (bi-)ordenable (véase también [Coh06, Corollary 1.5.10]). El problema general sigue aún abierto y puede extenderse al contexto más general de productos cruzados $E * G$ de un anillo de división E y un grupo ordenable a izquierda G, puesto que se sabe que son dominios (cf. [Sán08, Proposition 4.8]).

La conjetura fuerte de Atiyah para grupos libres de torsión sobre un subcuerpo K de $\mathbb{C}$ está íntimamente relacionada con este problema, y propone un candidato potencial a anillo de división. Más concretamente, si G es un grupo (numerable) libre de torsión y K es un subcuerpo de $\mathbb{C}$, podemos identificar $K[G]$ como subanillo del álgebra de grupos de von Neumann $\mathcal{N}(G)$ y del álgebra $\mathcal{U}(G)$ de operadores no acotados sobre $\ell^2(G)$ asociados (affiliated) a $\mathcal{N}(G)$. Este último anillo $\mathcal{U}(G)$ es un anillo von Neumann regular, y la conjetura fuerte de Atiyah plantea que la clausura de división de $K[G]$ en $\mathcal{U}(G)$ es un anillo de división. Observemos que esta conjectura es más fuerte que la conjetura de los divisores de cero de Kaplansky, puesto que ya implica que $K[G]$ es un dominio.

La conjetura fuerte de Atiyah (para un grupo arbitrario) surgió a partir de una pregunta de M.F. Atiyah en [Ati76] sobre la racionalidad de los L^2-números de Betti de una variedad con una G-acción propia cocompacta y libre, y al menos en la forma en que la presentaremos aquí, se suele atribuir a W. Lück y T. Schick (see [Lüc02, Chapter 10]). Existe gran cantidad de trabajos recientes que, de manera directa o indirecta, tratan este problema y sus variaciones, véase por ejemplo la siguiente lista (no exhaustiva) de artículos en torno a ella: [Lin93], [Sch00], [DLM+03], [LLS03], [FL06], [DL07], [LS07], [Lin08], [KLL09], [LOS12], [LS12], [Aus13], [Gra14], [Schr14], [AG17], [KLS17], [LL18], [Jai19]. La situación actual de esta conjetura, al menos en el momento de redacción de esta memoria, puede encontrarse en [Kam19].

Justo entre la familia de grupos (bi)-ordenables y la familia de grupos ordenables a izquierda se encuentra la familia de grupos localmente indicables ([Bro84], véase también [Nav10, Propositions 3.11 & 3.16] o [RR02, Theorem 4.1]). Recordemos que un grupo G es localmente indicable si todo subgrupo finitamente generado no trivial de G admite un homomorfismo sobreyectivo a $\mathbb{Z}$.

El problema principal y motivación original para esta tesis fue el estudio de la conjetura fuerte de Atiyah para esta familia de grupos, lo que a la postre permite zanjar el problema de embedding de Malcev para anillos de grupo $K[G]$ donde G es localmente indicable y K es un cuerpo de característica cero.

Con respecto a la pregunta 3', una particularidad interesante de esta familia de grupos es que, para todo producto cruzado $E*G$ de un anillo de división E y un grupo localmente indicable G, I. Hughes definió en [Hug70] un anillo de división, llamado hoy anillo de división de fracciones Hughes-free para $E * G$, y demostró que en caso de existir es único salvo un isomorfismo como en 2' (véase también [DHS04] o [Sán08, Hughes' theorem I]). Más aún, debido a las propiedades de $\mathcal{U}(G)$, el anillo propuesto por la conjetura fuerte de Atiyah como candidato a anillo de división es también el candidato principal para ser el anillo de división de fracciones Hughes-free para $K[G]$, donde K es subcuerpo de $\mathbb{C}$.

En general, el problema de existencia del anillo de división de fracciones Hughes-free para un producto cruzado de la forma anterior sigue aún abierto, y en el caso de existir, se desconoce si es siempre universal en el sentido de P. M. Cohn. No obstante, existen resultados recientes que apuntan en esta última dirección (véase [Jai20B]).

Un ejemplo particular sobre el que este problema está resuelto es el de los grupos libres: todo producto cruzado $E*F$ de un anillo de división E y un grupo libre F admite un anillo de división de fracciones universal porque es un fir, resultado usualmente atribuido a P. M. Cohn (véase [Lew69, Theorem I] o [Sán08, Theorem 4.22 (i)], y [Coh06, Corollary 7.5.14])), que puede demostrarse que además es Hughes-free ([Lew74, Proposition 6], véase también [Sán08, Example 6.19 & Proposition 6.23]). Más aún, en este anillo de división universal, toda matriz sobre $E * F$ para la que no exista un motivo "obvio" por el cual no pueda invertirse, es invertible.

Elaboremos con un poco más de detalle esta última frase. Dada la definición antes mencionada de rango en un anillo de división, si queremos que una matriz cuadrada A de tamaño $n \times n$ sobre R sea invertible en algún anillo de división, entonces A no puede admitir una descomposición de la forma $A = BC$ donde B, C sean matrices de tamaños $n \times k$ y $k \times n$, respectivamente, con $k < n$. El menor k para el que existe tal descomposición se denomina el rango interno $\rho(A)$ de A, y una matriz cuadrada de rango interno máximo se denomina plena. Podríamos preguntarnos entonces si existe un anillo de división en el que toda matriz plena (en particular, todo elemento) sobre R sea invertible, y en caso afirmativo es claro que tal anillo de división es el anillo de división universal de fracciones para R. La familia precisa de anillos para la que esto es posible fue estudiada por W. Dicks y E. D. Sontag en [DS78], y sus miembros recibieron el nombre de dominios de Sylvester porque son los anillos que satisfacen la ley de nulidad de Sylvester con respecto al rango interno: dadas matrices A y B de tamaños $n \times m$ y $m \times l$, respectivamente,

$$\rho(A) + \rho(B) \leq m + \rho(AB).$$

La familia de los firs (o anillos de ideales libres) introducida por P. M. Cohn en los años sesenta, anillos en que todo ideal a izquierda y todo ideal a derecha es libre con rango único, conforma una subfamilia de dominios de Sylvester. El contenido es además estricto, y el anillo de polinomios $K[x, y]$ en dos indeterminadas que conmutan entre sí con coeficientes en un cuerpo conmutativo K es un ejemplo de dominio de Sylvester ([DS78, Corollary 14], dado que $K[x, y] = (K[x])[y]$ es un álgebra libre en el conjunto $\{y\}$ sobre el dominio conmutativo de ideales principales $K[x]$) que no es un fir, puesto que el ideal (x, y) no es libre.

Continuando el razonamiento previo, si una matriz cuadrada A se vuelve invertible sobre un anillo de división, entonces ocurrirá lo propio con $A \oplus I_m$, la matriz diagonal por bloques con bloques A y la matriz identidad $m \times m$, I_m, para todo $m \geq 0$. Por tanto, no sólo A sino todas las matrices anteriores deben ser plenas sobre R, en cuyo caso se dice que A es establemente plena. En un dominio de Sylvester toda matriz plena es establemente plena, pero esta relación no es cierta en general y podríamos preguntarnos de nuevo si existe un anillo de división en el que toda matriz establemente plena sobre R se pueda invertir. Dada la necesidad de esta condición, si tal anillo de división existe debe ser universal. La familia de anillos que admiten un homomorfismo inyectivo a un anillo de división con estas características es la familia de dominios pseudo-Sylvester estudiada por P. M. Cohn y A. H. Schofield en [CS82]. Como ejemplo, $\mathcal{D}[x,y]$, el anillo de polinomios en dos indeterminadas que conmutan entre sí con coeficientes en un anillo de división no conmutativo $\mathcal{D}$, es un dominio pseudo-Sylvester ([CS82, Proposition 6.5]) que no es dominio de Sylvester porque admite un ideal proyectivo finitamente generado que no es libre (cf. [OS71, Proposition 1]), y los dominios de Sylvester son projective-free.

Tanto los dominios de Sylvester como los pseudo-Sylvester tienen dimensión débil a lo sumo 2, y dentro de la familia de grupos localmente indicables se sabe que toda extensión G de un grupo libre por $\mathbb{Z}$ (véanse grupos free-by-{infinite cyclic} o grupos de superficie) da lugar a productos cruzados $E*G$ de dimensión global a derecha e izquierda a lo sumo 2. Este hecho, junto con un criterio homológico desarrollado recientemente por A. Jaikin-Zapirain en [Jai20C] para identificar dominios de Sylvester, nos ha llevado a la búsqueda de dominios pseudo-Sylvester y de Sylvester entre estos productos cruzados, y de manera más general, entre productos cruzados de la forma $\mathfrak{F}*\mathbb{Z}$ donde $\mathfrak{F}$ es un fir.

Como ya comentamos, además de las preguntas 1' y 3', continuamos con el desarrollo de la teoría de funciones de rango de Sylvester sobre matrices, estudiando familias de anillos para las que se puede obtener una descripción del espacio de funciones de rango asociado, y analizando ejemplos particulares y construcciones relacionadas. Como comprobaremos a lo largo de los capítulos, las funciones de rango de Sylvester (en todas sus formas) no sólo dotan de una herramienta de clasificación (cf. [Mal80], [CS82], [Sch85, Capítulo 7], [Ele17]) sino de un lenguaje común con el que reformular y encarar diferentes problemas (cf. [AOP02], [Jai19], [Jai19S], [Jai20A], [Jai20B], [Jai20C], [JL20]), lo que las hace interesantes también como una entidad independiente (cf. [Goo91, Capítulos 16 y siguientes], [AC20], [Li20], [JiLi21]).

Además, en relación con la conjetura fuerte de Atiyah y los métodos empleados para abordarla, exploramos también otras conjeturas para la familia de grupos localmente indicables, a saber, las conjeturas de la independencia y del centro, y la conjetura fuerte del autovalor algebraico, todas ellas propuestas por A. Jaikin-Zapirain en [Jai19] y respondidas afirmativamente en el mismo artículo para grupos sóficos. Finalmente, abordamos también la conjetura de aproximación de Lück en el espacio de grupos marcados cuando el grupo que está siendo aproximado es virtualmente localmente indicable.

Resumen y conclusiones por capítulo

Demos una breve descripción de los temas considerados en cada capítulo y los principales resultados obtenidos.

En el **Capítulo 1**, primero introducimos y relacionamos algunas de las diferentes nociones de rango que se pueden encontrar en la literatura. Durante la introducción previa hablamos del *rango interno* de una matriz, y definimos los dominios de Sylvester como aquellos anillos para los que el rango interno satisface la ley de nulidad de Sylvester. De forma similar, un dominio pseudo-Sylvester es un anillo stably finite en el que se satisface la ley de nulidad con respecto al *rango estable* ρ^*, que se define para una matriz A como

$$\rho^*(A) = \lim_{m\to\infty} [\rho(A \oplus I_m) - m].$$

Las principales propiedades de estas dos nociones de rango en dominios de Sylvester y pseudo-Sylvester, respectivamente, sirven como punto de partida para introducir las *funciones de rango de Sylvester sobre matrices (o sobre módulos)* y sus propiedades básicas.

Para el caso particular de anillos von Neumann regulares, en los que todo elemento x tiene un "pseudo-inverso" (no necesariamente único) y tal que $xyx = x$, existe otra noción de rango, las llamadas *funciones de pseudo-rango* (véase [Goo91]), y mostramos que esta noción es equivalente a la noción de rango de Sylvester sobre matrices para esta familia de anillos.

Tras establecer las conexiones entre estos conceptos, pasamos a estudiar diversos escenarios en los que una función de rango de Sylvester definida en un anillo R puede "ser extendida", donde por "extensión" podemos hacer referencia a dos situaciones distintas. Por un lado, las funciones de rango de Sylvester sobre módulos se definen a priori sobre módulos finitamente presentados, y por tanto podríamos preguntar si es posible extenderlas a cualquier módulo. H. Li respondió afirmativamente a esta pregunta en [Li20] a través de la noción de rango de Sylvester bivariante sobre módulos. Por otro lado, si R fuese subanillo de un anillo S, podríamos preguntarnos bajo qué condiciones un rango en R puede extenderse a un rango en S. Será de particular interés el caso $S = R[t^{\pm 1}; \tau]$, el anillo de polinomios de Laurent asimétricos con coeficientes en R, donde τ es un automorfismo de R. Para este caso definiremos la extensión natural transcendente de un rango, variante de una de las nociones de extensión natural presentadas en [Jai19] y cuyo tratamiento ha sido generalizado y unificado en [JiLi21].

En el **Capítulo 2**, que está basado en [JL20B], estudiamos las propiedades básicas del espacio $\mathbb{P}(R)$ de funciones de rango de Sylvester definidas en un anillo R. Tras su introducción y un breve vistazo a los primeros ejemplos, analizamos en profundidad el espacio $\mathbb{P}(R)$ para familias de anillos particulares, esencialmente anillos de polinomios (de Laurent asimétricos) sobre anillos de división, y anillos que aparecen de forma natural como cocientes de éstos. La principal motivación para el estudio de estas familias fue la siguiente pregunta (cf. [Jai19S, Question 8.7]), que surgió durante un primer intento de A. Jaikin-Zapirain de demostrar el paso inductivo transcendente de su prueba de la conjetura de aproximación de Lück para grupos sóficos en [Jai19]. Dado un anillo R, usaremos $Z(R)$ para denotar su centro.

Pregunta 1. *Sea R un anillo simple von Neumann regular con una función de rango de Sylvester* rk *respecto de la cual es* rk*-completo. ¿Es cierto que toda función de rango de Sylvester sobre $Z(R)[t]$ se extiende de manera única a una función de rango de Sylvester sobre $R[t]$?*

Aquí, rk-completo significa que rk es fiel y que R es completo con respecto a la métrica δ_{rk} definida como $\delta_{\mathrm{rk}}(x, y) = \mathrm{rk}(x - y)$. Bajo las hipótesis anteriores, se puede probar que $\mathbb{P}(R) = \{\mathrm{rk}\}$ ([Goo91, Proposition 19.13 & Theorem 19.14]).

Los resultados principales de este capítulo pueden resumirse en las siguientes proposiciones y teoremas.

Proposición (Corollary 2.2.4). *Sea R un anillo primario artiniano a izquierda, y supongamos que existe un elemento $c \in Z(R)$ con orden de nilpotencia n tal que $J(R) = (c)$. Entonces toda función de rango sobre R está determinada por sus valores en c^i para $1 \leq i \leq n-1$, y los puntos extremos en $\mathbb{P}(R)$ son las funciones de rango de Sylvester sobre matrices $\mathrm{rk}_1, \dots, \mathrm{rk}_n$ definidas por*

$$\mathrm{rk}_k(c^i) = \begin{cases} \frac{k-i}{k} & \textit{si } i \leq k \\ 0 & \textit{en otro caso} \end{cases}$$

Cualquier otra función de rango se expresa de manera única como combinación convexa de $\mathrm{rk}_1, \dots, \mathrm{rk}_n$.

Supongamos ahora que R denota o bien a un dominio de Dedekind que no es un cuerpo, o a un anillo de polinomios de Laurent asimétrico $\mathcal{D}[t^{\pm 1};\tau]$ donde $\mathcal{D}$ es un anillo de división y τ es un automorfismo de $\mathcal{D}$ con orden interno finito (de manera que el anillo no es simple). Para cada ideal bilateral maximal $\mathfrak{m}$ de R y para todo entero positivo k, existe una función de rango de Sylvester sobre módulos $\dim_{\mathfrak{m},k}$ en R caracterizada por

$$\dim_{\mathfrak{m},k}(R/\mathfrak{n}^i) = \begin{cases} \frac{i}{k} & \text{si } \mathfrak{n} = \mathfrak{m} \text{ e } i \leq k \\ 1 & \text{si } \mathfrak{n} = \mathfrak{m} \text{ e } i > k \\ 0 & \text{si } \mathfrak{n} \neq \mathfrak{m} \end{cases}$$

para todo ideal bilateral maximal $\mathfrak{n}$ y entero positivo i. Sea $\dim_0$ la función de rango de Sylvester sobre módulos inducida por su anillo de división de Ore $\mathcal{Q}(R)$.

Teorema (Theorem 2.3.5, Theorem 2.5.8). *Sea R o bien un dominio de Dedekind que no es un cuerpo, o un anillo de polinomios de Laurent asimétrico $\mathcal{D}[t^{\pm 1};\tau]$ donde $\mathcal{D}$ es un anillo de división y τ es un automorfismo de $\mathcal{D}$ con orden interno finito. Los puntos extremos de $\mathbb{P}(R)$ son precisamente las funciones de rango $\dim_{\mathfrak{m},k}$ y $\dim_0$ antes definidas, y cualquier otra función de rango puede expresarse de manera única como una combinación convexa (posiblemente infinita) de éstas. Además, en el caso de polinomios de Laurent asimétricos, la inclusión $Z(R) \hookrightarrow R$ induce una biyección $\mathbb{P}(R) \to \mathbb{P}(Z(R))$.*

Como consecuencia de los resultados empleados en la demostración del teorema anterior, podemos dar respuesta afirmativa a la Pregunta 1 para el caso particular en que R es un anillo simple artiniano.

Proposición (Proposition 2.5.9). *Sea R un anillo simple artiniano. El homomorfismo inclusión $Z(R)[t] \hookrightarrow R[t]$ induce una biyección $\mathbb{P}(R[t]) \to \mathbb{P}(Z(R)[t])$. En particular, toda función de rango de Sylvester sobre $Z(R)[t]$ se extiende de manera única a una función de rango de Sylvester sobre $R[t]$.*

Cuando el automorfismo de $\mathcal{D}$ tiene orden interno infinito, $\mathcal{D}[t^{\pm 1};\tau]$ es un anillo simple y noetheriano a izquierda (y a derecha). Para esta familia de anillos, tenemos lo siguiente.

Proposición (Proposition 2.4.2). *En un anillo simple noetheriano a izquierda R existe una única función de rango de Sylvester sobre módulos, a saber, la inducida por su anillo clásico de cocientes a izquierda (simple y artiniano) $\mathcal{Q}_l(R)$.*

En el **Capítulo 3** introducimos en primer lugar la localización de Ore y la localización universal con el objetivo de describir brevemente los resultados principales de la teoría de Cohn sobre *anillos de división épicos.* Además, tras establecer los conceptos básicos asociados, recordamos la definición de los dos objetos universales con los que trataremos en las siguientes secciones y capítulos: el *anillo de división universal de fracciones*, y el *anillo de división de fracciones Hughes-free* para un producto cruzado $E * G$ de un anillo de división E y un *grupo localmente indicable* G. Asimismo, recordamos las caracterizaciones de *dominios de Sylvester y pseudo-Sylvester* en términos de su anillo de división universal de fracciones, desarrollamos un nuevo criterio homológico para identificar dominios pseudo-Sylvester (basado en la caracterización de dominios de Sylvester dada por A. Jaikin-Zapirain en [Jai20C]) y exploramos condiciones bajo las cuales un producto cruzado $\mathfrak{F} * \mathbb{Z}$, donde $\mathfrak{F}$ es un fir, es un dominio de Sylvester o pseudo-Sylvester.

Los resultados principales de este capítulo se obtuvieron como parte de un trabajo conjunto con F. Henneke en [HL20], y pueden resumirse de la siguiente manera.

Teorema (Theorem 3.5.9). *Sea $\mathfrak{F}$ un fir con $\mathfrak{F}$-anillo de división universal de fracciones $\mathcal{D}_{\mathfrak{F}}$, y consideremos un producto cruzado $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$. Entonces, las siguientes afirmaciones son ciertas:*

a) $\mathcal{S}$ es un dominio pseudo-Sylvester si y sólo si todo $\mathcal{S}$-módulo proyectivo finitamente generado es stably free.

b) $\mathcal{S}$ es un dominio de Sylvester si y sólo si es projective-free.

En cualquiera de las situaciones previas, el producto cruzado $\mathfrak{F} * \mathbb{Z}$ se extiende a un producto cruzado $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, y $\mathcal{D}_{\mathcal{S}} = \mathcal{Q}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$, su anillo de división de Ore, es el $\mathcal{S}$-anillo de división universal de fracciones. En particular, es isomorfo a la localización universal de $\mathcal{S}$ con respecto al conjunto de todas las matrices establemente plenas (resp. plenas).

Como aplicación particular del teorema previo, obtuvimos el siguiente resultado a través de los recientes avances en la conjetura de Farrell–Jones realizados por Bestvina–Fujiwara–Wigglesworth en [BFW19] y Brück–Kielak–Wu en [BKW19].

Teorema (Theorem 3.5.13). *Sea E un anillo de división y G un grupo obtenido como extensión*

$$1 \to F \to G \to \mathbb{Z} \to 1,$$

donde F es un grupo libre. Entonces todo producto cruzado $E * G$ es un dominio pseudo-Sylvester. En particular, $\mathcal{D}_{E*G} = \mathcal{Q}(\mathcal{D}_{E*F} * \mathbb{Z})$ es su $E * G$-anillo de división universal de fracciones, y es isomorfo a la localización universal de $E * G$ con respecto al conjunto de todas las matrices establemente plenas. Además, $E * G$ es un dominio de Sylvester si y sólo si posee la propiedad de cancelación stably free.

Ya era un hecho conocido que todo producto cruzado $E * G$ de un anillo de división E y un grupo G de la forma descrita en el teorema anterior admite un anillo de división de fracciones Hughes-free, y se ha probado además recientemente ([Jai20B, Theorem 3.7]) que en este caso este anillo de división es universal. De esta manera, el teorema previo supone una demostración independiente de la existencia del $E * G$-anillo de división universal de fracciones, y además identifica el conjunto concreto de matrices que se vuelven invertibles en este anillo de división.

Basándonos en el hecho de que el anillo de división Hughes-free es también universal para esta familia de grupos, en el resultado principal de [Grä20] y en la veracidad de la conjetura fuerte de Atiyah en este caso (véase [Lin93]), podemos dar descripciones explícitas de $\mathcal{D}_{E*G}$. Antes de enunciar el resultado notemos que, dado que todos los grupos G de la familia anterior son localmente indicables, podemos definir un orden de Conrad a izquierda $\leq$ en G. Denotemos por $E((G, \leq))$ al espacio de series de Malcev-Neumann, es decir, al E-espacio vectorial de series de potencias formales indexadas por G con coeficientes en E y soporte bien ordenado con respecto a $\leq$.

Teorema (Proposition 3.4.26, Corollary 3.5.14, Corollary 4.4.5). *En el contexto del teorema anterior, el $E * G$-anillo de división universal de fracciones es isomorfo al anillo de división de Dubrovin, es decir, a la clausura de división de $E * G$ en* $\mathrm{End}(E((G, \leq)))$, *donde $\leq$ es un orden de Conrad a izquierda en G. Si $E = K$ es un subcuerpo de $\mathbb{C}$, el $K[G]$-anillo de división universal de fracciones es también isomorfo al anillo de división de Linnell, es decir, a la clausura de división de $K[G]$ en $\mathcal{U}(G)$.*

Los últimos dos capítulos están basados en [JL20]. En el **Capítulo 4** introducimos los ingredientes necesarios para enunciar y demostrar la conjetura fuerte de Atiyah para grupos localmente indicables, a saber, la teoría de *anillos $*$-regulares épicos* presentada en [Jai19], las propiedades básicas del álgebra $\mathcal{U}(G)$ de operadores no acotados sobre $\ell^2(G)$ asociados (affiliated) al álgebra de grupos de von Neumann $\mathcal{N}(G)$, y la teoría de *semianillos racionales* desarrollada en [DHS04] y [Sán08]. El método inductivo basado en la noción de complejidad construida en estas últimas referencias, junto con el hecho de que $\mathcal{U}(G)$ admite una función de rango de Sylvester fiel rk_G que satisface el análogo a la condición Hughes-free para anillos de división, nos permite probar que la clausura de división de $\mathbb{C}[G]$ en $\mathcal{U}(G)$ es el $\mathbb{C}[G]$-anillo de división de fracciones Hughes-free, demostrando en particular la conjetura fuerte de Atiyah para esta familia de grupos. Los resultados principales del capítulo son los siguientes.

Teorema (Theorem 4.4.2, Corollary 4.4.3). *Sean G un grupo numerable localmente indicable y K un subcuerpo de $\mathbb{C}$. Entonces la conjetura fuerte de Atiyah sobre K es cierta y la clausura de división $\mathcal{D}_{K[G]}$ de $K[G]$ en $\mathcal{U}(G)$ es el $K[G]$-anillo de división de fracciones Hughes-free.*

Gracias a las sugerencias de F. Henneke y D. Kielak, probamos también un resultado de estabilidad para la conjetura fuerte de Atiyah en este contexto.

Proposición (Proposition 4.4.6). *Sean K un subcuerpo de $\mathbb{C}$ y G_2 un grupo numerable construido como extensión*

$$1 \to G_1 \to G_2 \to G_3 \to 1,$$

donde G_1 es un subgrupo normal libre de torsión de G_2 y G_3 es un grupo localmente indicable. Si G_1 satisface la conjetura fuerte de Atiyah sobre K, entonces G_2 también.

Finalmente, el **Capítulo 5** está dedicado a explorar las consecuencias de los resultados previos sobre la conjetura fuerte de Atiyah y otras conjeturas relacionadas que pueden abordarse por medio de las mismas técnicas. En [Jai19], A. Jaikin-Zapirain formuló tres conjeturas en relación al ya mencionado rango rk_G en $\mathcal{U}(G)$ y al objeto central de su artículo, es decir, la clausura $*$-regular $\mathcal{R}_{K[G]}$ de $K[G]$ en $\mathcal{U}(G)$ (para un subcuerpo K de $\mathbb{C}$ cerrado bajo conjugación compleja). Estas tres conjeturas fueron confirmadas en el mismo artículo para grupos sóficos, y aquí las probamos para grupos localmente indicables.

La *conjetura de la independencia* plantea si, para un cuerpo K que admite más de un homomorfismo inyectivo a $\mathbb{C}$ y para una matriz A sobre $K[G]$, el rk_G-rango de la imagen de A es independiente del homomorfismo. Y en efecto, la conjetura se satisface si G es localmente indicable.

Proposición (Proposition 5.1.1). *Sean G un grupo numerable localmente indicable, K un cuerpo de característica cero y $\varphi_1, \varphi_2 : K \to \mathbb{C}$ dos homomorfismos inyectivos de K en $\mathbb{C}$. Entonces, para toda matriz $A \in \mathrm{Mat}_{n\times m}(K[G])$,*

$$\mathrm{rk}_G(\varphi_1(A)) = \mathrm{rk}_G(\varphi_2(A))$$

En realidad la hipótesis de numerabilidad puede obviarse, dado que rk_G se puede definir para grupos arbitrarios y la demostración, con leves modificaciones, sigue siendo válida. Con esta proposición y el resultado principal del Capítulo 4 demostramos la existencia de anillos de división de fracciones Hughes-free para anillos de grupo sobre cuerpos de característica cero.

Corolario (Corollary 5.1.2). *Sean G un grupo localmente indicable y K un cuerpo de característica cero. El anillo de grupo $K[G]$ admite un anillo de división de fracciones Hughes-free.*

La *conjetura fuerte del autovalor algebraico* plantea, por su parte, la algebraicidad de los posibles autovalores complejos de una matriz A sobre $\mathcal{R}_{K[G]}$ o, de manera más general, sobre la clausura de división $\mathcal{D}_{K[G]}$.

Proposición (Proposition 5.1.5). *Sean G un grupo numerable localmente indicable y K un subcuerpo de $\mathbb{C}$. Entonces, para todo $\lambda \in \mathbb{C}$ que no es algebraico sobre K y para toda $A \in \mathrm{Mat}_n(\mathcal{D}_{K[G]})$, la matriz $A - \lambda I$ es invertible en $\mathcal{U}(G)$.*

Se sabe que el álgebra de grupos de von Neumann $\mathcal{N}(G)$ de un grupo numerable ICC es un factor, en el sentido de que $Z(\mathcal{N}(G)) = \mathbb{C}$, y que además esta propiedad se extiende a $\mathcal{U}(G)$, es decir, $Z(\mathcal{U}(G)) = \mathbb{C}$. La *conjetura del centro* plantea que esta propiedad es también cierta si sustituimos $\mathcal{N}(G)$ por la completación de $\mathcal{R}_{K[G]}$ con respecto a la rk_G-métrica descrita en el resumen del Capítulo 2. Para un grupo localmente indicable, $\mathcal{R}_{K[G]}$ ya es rk_G-completo, puesto que la conjetura fuerte de Atiyah implica que $\mathcal{R}_{K[G]}$ es un anillo de división y que rk_G toma valores enteros sobre $\mathcal{R}_{K[G]}$. En esta memoria demostramos el siguiente resultado, ligeramente más general, sustituyendo $\mathcal{R}_{K[G]}$ por la clausura de división $\mathcal{D}_{K[G]}$.

Proposición (Proposition 5.1.6, Corollary 5.1.7). *Sean G un grupo numerable localmente indicable, K un subcuerpo de $\mathbb{C}$ y $\mathcal{D}_{K[G]}$ la clausura de división de $K[G]$ en $\mathcal{U}(G)$. Entonces*

$$\mathcal{D}_{K[G]} \cap \mathbb{C} = K.$$

En particular, si G es ICC, $Z(\mathcal{D}_{K[G]}) = K$.

Tras analizar estas tres conjeturas, pasamos a considerar la *conjetura de aproximación de Lück en el espacio de grupos marcados.* Una de las formas originales de esta conjetura de aproximación plantea que, para todo CW-complejo conexo y compacto X con grupo fundamental G y para toda cadena anidada de subgrupos normales $\{G_i\}$ de G con $\bigcap G_i = \{1\}$, se pueden aproximar los L^2-números de Betti del recubrimiento universal $\tilde{X}$ a partir de los L^2-números de Betti de los recubrimientos $\bar{X}_i$ de X asociados a la cadena (cf. [Kam19, Conjecture 1.7]), o sea, que para todo $k \geq 0$,

$$\lim_{i\to\infty} b_k^{(2)}(\bar{X}_i) = b_k^{(2)}(\tilde{X}).$$

La conjetura fue resuelta por W. Lück en [Lüc94] para el caso en que G es residualmente finito, y desde entonces se han estudiado varias reformulaciones y generalizaciones de la misma.

En particular, la conjetura se puede formular en términos de la función de rango rk_G inducida por $\mathcal{U}(G)$ en el anillo de grupo $K[G]$ y las funciones de rango asociadas a alguna "aproximación" de G. Por ejemplo, en [Jai19] se demuestra la veracidad de la conjetura cuando la aproximación de G considerada es sófica. Aquí probamos su veracidad cuando se consideran aproximaciones en el espacio de grupos marcados y el grupo G aproximado es virtualmente localmente indicable. De forma más precisa, probamos lo siguiente:

Teorema (Theorem 5.2.13). *Sean F un grupo libre finitamente generado y $\{M_i\}_{i\in\mathbb{N}}$ una sucesión que converge a M en el espacio de grupos marcados $\mathrm{MG}(F)$. Denotemos $G_i = F/M_i$, $G = F/M$, y sean $\pi_G : \mathbb{C}[F] \to \mathbb{C}[G]$ y $\pi_{G_i} : \mathbb{C}[F] \to \mathbb{C}[G_i]$ los homomorfismos inducidos. Si G es virtualmente localmente indicable, entonces para toda matriz A sobre $\mathbb{C}[F]$,*

$$\lim_{i\to\infty} \mathrm{rk}_{G_i}(\pi_{G_i}(A)) = \mathrm{rk}_G(\pi_G(A)).$$

Concluimos el capítulo aplicando las técnicas del Capítulo 4 a la cuestión de universalidad del anillo de división de fracciones Hughes-free. No pudimos probar que este último (siempre que exista) sea universal en el sentido de P. M. Cohn, pero pudimos probar la siguiente proposición.

Proposición (Corollary 5.3.3). *Sea $E * G$ un producto cruzado de un anillo de división E y un grupo localmente indicable G. Si existen tanto el $E * G$-anillo de división de fracciones Hughes-free como el universal, entonces son $E * G$-isomorfos.*

En particular, si G es numerable, $E = K$ es un subcuerpo de $\mathbb{C}$ y existe el $K[G]$-anillo de división universal de fracciones, entonces es isomorfo a la clausura de división de $K[G]$ en $\mathcal{U}(G)$.

Introduction and conclusions

Although this may not be chronologically accurate in the development of the thesis, the main motivations to this work can be explained from the perspective of a first course in linear algebra and ring theory.

Namely, one of the first algebraic structures that we encounter when being introduced to these topics is that of a (commutative) field, usually having $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ as particular instances. The facts that every non-zero element in a field K has a multiplicative inverse and that every finitely generated K-module, i.e., vector space, is free with a fixed finite number of elements in any basis makes them a desirable framework to start with.

As a consequence, given a commutative ring R, it is natural to ask about the possibility of finding or constructing an overring which is a field, so that we can still make use of the linear algebra machinery and inherit some of its basic properties. For instance, the ring of integers $\mathbb{Z}$ is naturally a subring of $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Hence, there are three vague questions that can come up to our minds at this point:

1. Given a commutative ring R, can we embed R into a field K?

2. If, for a particular ring R, the answer to the first question is positive, is the field K "unique"?

3. If, after making precise what uniqueness means, the answer to the second question is negative, is there a way to characterize (some of) the fields in which R embeds? Is there a "universal" one?

Let us in the following update and answer these questions in the commutative case. An unavoidable obstacle to achieve goal 1. is the existence of non-trivial zero-divisors in the ring, i.e., of non-zero elements a and b with $ab = 0$ in R; however, when we are not in this situation, i.e., when we deal with a domain, we are taught to construct the field of fractions associated to R, a field whose description resembles that of $\mathbb{Q}$ with respect to $\mathbb{Z}$, namely, its elements are fractions with numerator and non-zero denominator in R together with the usual operations. Therefore, the answer to question 1. is positive if and only if R is a domain.

We have already introduced an example in which there are multiple choices of fields (in fact, an uncountable number of choices if we consider field extensions of $\mathbb{Q}$), but some of them can either be too "large" or too "complicated" in comparison with the original ring $\mathbb{Z}$. In this sense, the construction of $\mathbb{Q}$ from $\mathbb{Z}$ makes it more recognizable and treatable: we just add an inverse $\frac{1}{b}$ for any non-zero integer b and then the strictly necessary elements (fractions $\frac{a}{b}$) in order to define a ring structure on the new resulting set. We say that $\mathbb{Q}$ is generated by $\mathbb{Z}$, since each rational number can be obtained from integers by performing sums, substractions, products and inversions. In view of this discussion, and since we may enlarge a field to obtain bigger and bigger fields containing the original one, it seems reasonable to restrict question 2. to the case of fields that are generated in the previous sense by R, and the uniqueness should be considered up to an isomorphism making the following commute

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ K \arrow[rr, "\cong"] & & K'. \end{tikzcd}$$

Moreover, observe that as in the case of $\mathbb{Z}$ and $\mathbb{Q}$, the field of fractions $\mathcal{Q}(R)$ of R contains the least elements needed to form a field starting from R, and hence it is contained in any other field containing R. More precisely, any embedding of R into a field K can be extended to an embedding of $\mathcal{Q}(R)$ into K. Thus, $\mathcal{Q}(R)$ is, up to an isomorphism as before, the unique field with the properties that contains and it is generated by R.

Summing up, the answers to the previous questions are the following.

1. R can be embedded into a field if and only if R is a domain.

2. If R is a domain, its field of fractions $\mathcal{Q}(R)$ is the unique field (up to an isomorphism as before) in which R embeds and that is generated by R as field.

3. Question 2. is already positive.

Therefore, every question has, in the commutative setting, a satisfactory answer once we specify the kind of fields we want to deal with, namely, fields generated by the original ring.

In the non-commutative setting, the analog of the notion of field is the notion of division ring (or skew field) $\mathcal{D}$. As in the case of a field, every non-zero element in a division ring has a multiplicative inverse and every finitely generated left (and right) $\mathcal{D}$-module is free with a fixed finite number of elements in any basis. Moreover, except for those that rely on the commutativity of the product, most of the properties of fields carry over to division rings, and hence, in analogy with the commutative case, it is natural to ask whether given a non-commutative ring there exists a division ring in which it embeds. Taking into account the answers for commutative rings, one could come up with the following questions:

1'. Given a non-commutative domain R, can we embed R into a division ring $\mathcal{D}$?

2'. Assume that the answer to the first question is positive for R, and that $\mathcal{D}$ is generated by R as a division ring. Is $\mathcal{D}$ unique? i.e., if there exists another division ring $\mathcal{D}'$ in which R embeds and which is generated by R, does there exist an isomorphism making the diagram

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ \mathcal{D} \arrow[rr, "\cong"] & & \mathcal{D}'. \end{tikzcd}$$

commute?

3'. If the answer to the second question is negative, is there a way to characterize (some of) the division rings in which R embeds? Is there a "universal" one?

Unfortunately, in this case the answers to all of these questions are not as satisfactory. Regarding the first one, A. I. Malcev proved in [Mal37] that there exist domains which cannot be embedded into division rings (see also [Coh06, Exercises 2.11, 9]), while regarding the second question there exist domains R that can be embedded in many non-isomorphic division rings generated by R (see, for instance, [Fis71], or [Sán08, Corollary 7.13]).

However, despite the wildness of the questions for non-commutative rings, the theory of epic division rings developed by P. M. Cohn in the 70's (see [Coh06, Chapter 7]) answers abstractly questions 1' and 2'. This theory deals with the more general situation of finding homomorphisms (not necessarily injective) from a ring R (not necessarily a domain) to division rings $\mathcal{D}$ such that $\mathcal{D}$ is generated by the image of R (this is precisely what the adjective "epic" means in this context). He characterized the latter, up to an isomorphism as in 2', in terms of the matrices over R whose image in $\mathcal{D}$ is invertible. In this sense, he defined the universal epic division ring for R to be the epic division ring $\mathcal{U}$ with the property that, if a matrix A over R can be inverted in some division ring, then it is also invertible in $\mathcal{U}$. Of course, there is no reason for such a division ring to exist in general.

P. M. Cohn's characterization of epic division rings in terms of matrices is also an analog of the general commutative situation. For a commutative ring R, a homomorphism $\phi : R \to K$ to an epic field K is completely characterized by its kernel $\ker \phi$, which is always a prime ideal of R. More precisely, if we are given a pair (K, ϕ) as before, then K is isomorphic to the residue field of the local ring $R_{\ker \phi}$, the localization of R at the prime ideal $\ker \phi$, and two different prime ideals give rise by this procedure to non-isomorphic epic fields, since the elements of R becoming invertible in each of them (i.e., the ones outside the prime ideal) are different. This gives a bijective correspondence between the set of epic fields up to isomorphism and the set of prime ideals of the ring.

In the same spirit, P. M. Cohn defined the notion of prime matrix ideal and showed that there is a bijective correspondence between epic division R-rings $(\mathcal{D}, \phi)$ (up to the isomorphism defined in 2') and prime matrix ideals of R. In analogy to the previous case, given an epic division R-ring $(\mathcal{D}, \phi)$, the collection of square matrices mapping to singular matrices over $\mathcal{D}$, the so-called singular kernel of ϕ, forms a prime matrix ideal $\mathcal{P}$ of R, and one can recover $\mathcal{D}$ from this information by localizing R at $\mathcal{P}$ (what can be done from a presentation of R by formally adding inverses to the square matrices outside $\mathcal{P}$) to obtain a non-zero local ring $R_{\mathcal{P}}$ whose residue division ring is (isomorphic to) $\mathcal{D}$.

Note that no new object is constructed in this way for a commutative ring, since a square matrix A over a commutative ring R becomes invertible under $\phi : R \to K$ if and only if the determinant of $\phi(A)$, which is an element in $\phi(R)$, is non-zero, and hence the information about the singular kernel is already enclosed in $\ker \phi$.

P. Malcolmson reformulated in [Mal80] this characterization of epic division R-rings in many different ways. One of them, namely algebraic rank functions (nowadays known as Sylvester matrix rank functions) will arise as the central tool and language during this report. More concretely, as in the case of commutative fields, in a division ring $\mathcal{D}$ one can define the rank $\mathrm{rk}_{\mathcal{D}}(A)$ of an $n \times m$ matrix A, for instance, as the minimum non-negative integer k such that A admits a decomposition $A = BC$ with B of size $n \times k$ and C of size $k \times m$. Moreover, a square matrix A is invertible over $\mathcal{D}$ if and only if it has maximum rank. Thus, a homomorphism $\phi : R \to \mathcal{D}$ induces an "integer-valued rank function" on R by defining the rank of a matrix over R as the $\mathrm{rk}_{\mathcal{D}}$-rank of its image under ϕ. In this manner, the square matrices becoming invertible are precisely the ones with maximum rank, and the complement (in the set of square matrices over R) of this set is the singular kernel of the homomorphism.

The rank over a division ring can be uniquely characterized in terms of some of its properties, thus allowing us to axiomatize and formalize the notion of Sylvester matrix rank function on a ring R (see Chapter 1). P. M. Cohn's result may be restated in this language as follows.

Theorem (Cohn, Malcolmson). *There exists a bijective correspondence between the collection of epic division R-rings up to isomorphism and the collection of integer-valued Sylvester matrix rank functions on R.*

Throughout this document, we consider particular instances of questions 1' and 3', and problems focused on a better understanding of the space of Sylvester rank functions associated to a ring R.

With respect to question 1', one open problem usually referred to as Malcev's embedding problem ([KM18, Problem 1.6]), asks whether the answer is positive for group rings $K[G]$, where K is a commutative field and G is a left orderable group. A.I. Malcev ([Mal48]) and B. H. Neumann ([Neu49]) proved independently that this is the case if the group G is (bi-)ordered (see also [Coh06, Corollary 1.5.10]). The general question is still open and can be extended to the more general context of crossed products $E * G$ of a division ring E and a left orderable group G, since these are known to be domains (cf. [Sán08, Proposition 4.8]).

Deeply related to this problem is the strong Atiyah conjecture for a torsion-free group over a subfield K of $\mathbb{C}$, which goes a step further and proposes a potential candidate for the division ring. More precisely, if G is a torsion-free (countable) group and K is a subfield of $\mathbb{C}$, one can embed $K[G]$ into the group von Neumann algebra $\mathcal{N}(G)$ and the algebra $\mathcal{U}(G)$ of unbounded operators on $\ell^2(G)$ affiliated to $\mathcal{N}(G)$. The latter ring $\mathcal{U}(G)$ is von Neumann regular, and the strong Atiyah conjecture states that the division closure of $K[G]$ inside $\mathcal{U}(G)$ is a division ring. Observe that this conjecture is stronger than Kaplansky's zero-divisor conjecture, since it already implies that $K[G]$ is a domain.

The strong Atiyah conjecture (for arbitrary groups) arose from a question of M.F. Atiyah in [Ati76] about the rationality of the L^2-Betti numbers of a manifold with a cocompact free proper G-action, and at least in the form that we present here, the strong Atiyah conjecture is usually attributed to W. Lück and T. Schick (see [Lüc02, Chapter 10]). There is a great recent body of work about and around the strong Atiyah conjecture and its variants, see for instance the following non-comprehensive list of articles: [Lin93], [Sch00], [DLM$^+$03], [LLS03], [FL06], [DL07], [LS07], [Lin08], [KLL09], [LOS12], [LS12], [Aus13], [Gra14], [Schr14], [AG17], [KLS17], [LL18], [Jai19]. The state of the art up to the time of writing can be found in [Kam19].

Right in the middle between the family of (bi-)orderable groups and the family of left orderable groups is the family of locally indicable groups ([Bro84], see also [Nav10, Propositions 3.11 & 3.16] or [RR02, Theorem 4.1]). Recall that a group G is locally indicable if every finitely generated non-trivial subgroup of G admits a surjective homomorphism onto $\mathbb{Z}$.

The main problem and original motivation to the thesis was the verification of the strong Atiyah conjecture for this family of groups, what a posteriori settles Malcev's embedding problem for group rings of locally indicable groups with coefficients in a characteristic zero field.

With regard to question 3', one particularly interesting feature of this family of groups is that, for any crossed product $E * G$ of a division ring E and a locally indicable group G, I. Hughes defined in [Hug70] a division ring, now called Hughes-free division ring of fractions for $E * G$, and showed that provided it exists it is unique up to an isomorphism as in 2' (see also [DHS04] or [Sán08, Hughes' theorem I]). Moreover, because of the properties of $\mathcal{U}(G)$, the ring proposed by the strong Atiyah conjecture to be a division ring in which $K[G]$ embeds is also the candidate to be the Hughes-free division ring of fractions for $K[G]$.

In general, it is still an open question whether a crossed product $E * G$ admits a Hughes-free division ring of fractions, and in the case it admits one, it is unclear whether it is always universal in the sense of P. M. Cohn. However, there are examples of groups within this family in which the Hughes-free division ring of fractions is known to exist and to be universal (see [Jai20B]).

This is the case of free groups, i.e., every crossed product $E * F$, where E is a division ring and F is a free group, admits a universal division ring of fractions since they are firs, a result which is commonly attributed to P. M. Cohn (see [Lew69, Theorem I] or [Sán08, Theorem 4.22 (i)], and [Coh06, Corollary 7.5.14]), and this can be shown to be Hughes-free ([Lew74, Proposition 6], see also [Sán08, Example 6.19 & Proposition 6.23]). Moreover, in this universal division ring of fractions, every matrix over $E * F$ without an "obvious" obstruction to become invertible, actually becomes invertible.

Let us elaborate a little around the last sentence. Given the definition of rank on a division ring that we mentioned earlier, if an $n \times n$ square matrix A over a ring R is to become invertible in some division ring, it cannot admit a decomposition $A = BC$

for matrices B, C of sizes $n \times k$ and $k \times n$ with $k < n$. The least k for which such a decomposition exists is called the inner rank $\rho(A)$ of A, and a square matrix with maximum inner rank is said to be full. One can then wonder whether there exists a division ring in which every full matrix (in particular every element) over R becomes invertible, and in the affirmative case this division ring is clearly the universal division ring of fractions for R. The precise family of rings for which this is possible was first studied by W. Dicks and E. D. Sontag in [DS78], and received the name of Sylvester domains because they are the rings satisfying Sylvester's law of nullity for the inner rank: given matrices A and B of sizes $n \times m$ and $m \times l$, respectively,

$$\rho(A) + \rho(B) \leq m + \rho(AB).$$

The family of firs (or free ideal rings) introduced by P. M. Cohn in the 60's, rings in which every left and every right ideal is free of unique rank, forms a subfamily of Sylvester domains. The containment is strict and the polynomial ring $K[x, y]$ in two commuting indeterminates with coefficients in a (commutative) field K is an example of Sylvester domain ([DS78, Corollary 14], since $K[x, y] = (K[x])[y]$ is a free algebra on $\{y\}$ over the commutative principal ideal domain $K[x]$) which is not a fir, since the ideal (x, y) is not free.

Continuing the previous reasoning, if a square matrix A becomes invertible over a division ring, then $A \oplus I_m$, the block diagonal matrix with blocks A and the $m \times m$ identity matrix I_m, is also invertible for every $m \geq 0$. Thus, not only A but all these matrices should be full in R, and in this case A is said to be stably full. In a Sylvester domain every full matrix is already stably full, but this relation does not hold in general and we may again wonder whether there exists a division ring in which every stably full matrix over R can be inverted. Given the necessity of this condition, if such a division ring exists it must be universal. The family of rings that can be embedded in a division ring with this description is the family of pseudo-Sylvester domains studied by P. M. Cohn and A. H. Schofield in [CS82]. For instance, the polynomial ring $\mathcal{D}[x, y]$ in two commuting indeterminates with coefficients in a non-commutative division ring $\mathcal{D}$ is a pseudo-Sylvester domain ([CS82, Proposition 6.5]) which cannot be a Sylvester domain because it admits a finitely generated non-free projective ideal (cf. [OS71, Proposition 1]) and the latter are projective-free.

Both Sylvester and pseudo-Sylvester domains are known to have weak dimension at most 2, and within the family of locally indicable groups, every free-by-{infinite cyclic} group or surface group G is known to even satisfy that every crossed product $E * G$ with a division ring E has right and left global dimension at most 2. This fact, together with a recent homological criterion developed by A. Jaikin-Zapirain in [Jai20C] to identify Sylvester domains, led us to the search of pseudo-Sylvester and Sylvester domains among these crossed products, and more generally, among crossed products of the form $\mathfrak{F} * \mathbb{Z}$ where $\mathfrak{F}$ is a fir.

As we already mentioned, besides questions 1' and 3', we also develop further the theory of Sylvester matrix rank functions, studying examples of rings for which a description of the space of Sylvester matrix rank functions can be achieved, and analyzing particular instances and associated constructions. As we shall find out throughout the chapters, Sylvester rank functions (in their various forms) provide not only a classifying tool (cf. [Mal80], [CS82], [Sch85, Chapter 7], [Ele17]) but a common language to rephrase and address different problems (cf. [AOP02], [Jai19], [Jai19S], [Jai20A], [Jai20B], [Jai20C], [JL20]), and this makes them interesting also as an independent entity (cf. [Goo91, Chapters 16 and following], [AC20], [Li20], [JiLi21]).

Furthermore, in relation with the strong Atiyah conjecture and the methods used to tackle it, we also explore other conjectures for the family of locally indicable groups, namely, the independence, the center and the strong algebraic eigenvalue conjectures posed by A. Jaikin-Zapirain in [Jai19] and solved in the positive in the same paper for sofic groups, and Lück's approximation conjecture in the space of marked groups.

Summary and conclusions by chapter

Let us give a brief description of the topics considered in each chapter and the main results obtained.

In **Chapter 1**, we first introduce and relate some of the different notions of rank that appear in the literature. We already talked during the previous introduction about the *inner rank* of a matrix, and defined Sylvester domains as the family of rings for which the inner rank satisfies Sylvester's law of nullity. Similarly, pseudo-Sylvester domains are defined as stably (or weakly) finite rings in which the law of nullity is satisfied with respect to the *stable rank* ρ^*, given for a matrix A by

$$\rho^*(A) = \lim_{m\to\infty} [\rho(A \oplus I_m) - m].$$

The main properties of these two notions of rank on Sylvester and pseudo-Sylvester domains, respectively, serve as a starting point for the introduction of *Sylvester matrix (and module) rank functions*, whose basic properties are studied thereafter.

For the particular case of von Neumann regular rings, on which every element x has a (non-necessarily unique) "pseudo-inverse" y satisfying $xyx = x$, there exists another notion of rank, namely, *pseudo-rank functions* (see [Goo91]). This notion is shown to be equivalent to the notion of Sylvester matrix rank function for this family of rings.

After establishing the connections between these notions, we study several scenarios in which a given Sylvester rank function on a ring R can be "extended", where "extending" may have two different interpretations. On the one hand, Sylvester module rank functions are defined a priori on finitely presented modules over the ring, and hence one can ask whether they can be further extended to any module. H. Li completely settled the question by proving in [Li20], through the notion of bivariant Sylvester module rank functions, that this can always be achieved. On the other hand, one may also look for conditions under which the rank on R extends to some overring S. It shall be of particular interest the definition and characterizations of the natural transcendental extension of a rank rk on R to a skew Laurent polynomial ring $R[t^{\pm 1}; \tau]$, where τ is an automorphism of R. The notions of natural extensions were introduced in [Jai19], while a more general and unifying treatment on the topic is given in [JiLi21].

In **Chapter 2**, which is based on [JL20B], we study the basic properties of the space $\mathbb{P}(R)$ of Sylvester rank functions defined on a ring R. After its introduction and the first examples, we analyze further the space $\mathbb{P}(R)$ for particular families of rings, which are essentially (skew Laurent) polynomial rings over division rings and rings that appear naturally as their quotients. The main motivation for this was the following question (cf. [Jai19S, Question 8.7]), that arose during a first attempt of A. Jaikin-Zapirain to prove the transcendental inductive step in his proof of the sofic Lück's approximation conjecture in [Jai19]. For a ring R, we use $Z(R)$ to denote its center.

Question 1. *Let R be a simple von Neumann regular ring with a Sylvester rank function* rk *such that R is* rk*-complete. Is it true that every Sylvester rank function on $Z(R)[t]$ extends uniquely to a Sylvester rank function on $R[t]$?*

Here, rk-complete means that rk is faithful and that R is complete with respect to the metric δ_{rk} defined by $\delta_{\mathrm{rk}}(x, y) = \mathrm{rk}(x - y)$. Under the above hypothesis, it can be shown that $\mathbb{P}(R) = \{\mathrm{rk}\}$ ([Goo91, Proposition 19.13 and Theorem 19.14]).

The main results of this chapter can be summarized in the following propositions and theorems.

Proposition (Corollary 2.2.4)**.** *Let R be a left artinian primary ring, and assume that there exists an element $c \in Z(R)$ with order of nilpotency n such that $J(R) = (c)$. Then any rank function on R is determined by its values on c^i for $1 \le i \le n-1$, and the extreme points in $\mathbb{P}(R)$ are the Sylvester matrix rank functions $\mathrm{rk}_1, \dots, \mathrm{rk}_n$ defined by*

$$\mathrm{rk}_k(c^i) = \begin{cases} \frac{k-i}{k} & \text{if } i \le k \\ 0 & \text{otherwise} \end{cases}$$

Any other rank function can be uniquely expressed as a convex combination of them.

Now, let R denote either a Dedekind domain which is not a field, or a skew Laurent polynomial ring $\mathcal{D}[t^{\pm 1}; \tau]$ where $\mathcal{D}$ is a division ring and τ is an automorphism of finite inner order (so that the ring is not simple). For each maximal two-sided ideal $\mathfrak{m}$ of R and every positive integer k, there exists a Sylvester module rank function $\dim_{\mathfrak{m},k}$ on R characterized by

$$\dim_{\mathfrak{m},k}(R/\mathfrak{n}^i) = \begin{cases} \frac{i}{k} & \text{if } \mathfrak{n} = \mathfrak{m} \text{ and } i \le k \\ 1 & \text{if } \mathfrak{n} = \mathfrak{m} \text{ and } i > k \\ 0 & \text{if } \mathfrak{n} \neq \mathfrak{m} \end{cases}$$

for every maximal two-sided ideal $\mathfrak{n}$ and positive integer i. Let $\dim_0$ denote the Sylvester module rank function induced by the Ore division ring $\mathcal{Q}(R)$.

Theorem (Theorem 2.3.5, Theorem 2.5.8)**.** *Let R denote either a Dedekind domain which is not a field, or a skew Laurent polynomial ring $\mathcal{D}[t^{\pm 1}; \tau]$ where $\mathcal{D}$ is a division ring and τ is an automorphism of finite inner order. The extreme points on $\mathbb{P}(R)$ are precisely the rank functions $\dim_{\mathfrak{m},k}$ and $\dim_0$ defined above, and any other rank function can be uniquely expressed as a (possibly infinite) convex combination of them. Moreover, in the skew Laurent polynomial case, the inclusion map $Z(R) \hookrightarrow R$ defines a bijection $\mathbb{P}(R) \to \mathbb{P}(Z(R))$.*

As a consequence of the results used for the proof of the previous theorem, we give a positive answer to Question 1 for the particular case of simple artinian rings.

Proposition (Proposition 2.5.9). *Let R be a simple artinian ring. The inclusion map $Z(R)[t] \hookrightarrow R[t]$ defines a bijection $\mathbb{P}(R[t]) \to \mathbb{P}(Z(R)[t])$. In particular, every Sylvester rank function on $Z(R)[t]$ can be uniquely extended to a Sylvester rank function on $R[t]$.*

In the case in which the automorphism of $\mathcal{D}$ has infinite inner order, $\mathcal{D}[t^{\pm 1};\tau]$ is a left (and right) noetherian simple ring. For this family of rings, we have the following.

Proposition (Proposition 2.4.2). *On a left noetherian simple ring R, there exists only one Sylvester module rank function, namely, the one induced by its (simple artinian) classical left quotient ring $\mathcal{Q}_l(R)$.*

In **Chapter 3** we begin by introducing Ore and universal localizations in order to give a brief description of the main results in Cohn's theory of *epic division rings*. Moreover, after setting the necessary background, we recall the definition of the two universal objects that we shall be dealing with in the subsequent sections and chapters: the *universal division ring of fractions*, and the *Hughes-free division ring of fractions* for a crossed product $E * G$ of a division ring and a *locally indicable group*. We recall the characterizations of *Sylvester and pseudo-Sylvester domains* in terms of their universal division ring of fractions, develop a new homological criterion for a ring to be a pseudo-Sylvester domain (based on the characterization of Sylvester domains given by A. Jaikin-Zapirain in [Jai20C]) and we explore conditions under which a crossed product $\mathfrak{F} * \mathbb{Z}$, where $\mathfrak{F}$ is a fir, is a Sylvester or pseudo-Sylvester domain.

The main results of this chapter were obtained as a joint work with F. Henneke in [HL20], and can be summarized as follows.

Theorem (Theorem 3.5.9). *Let $\mathfrak{F}$ be a fir with universal division $\mathfrak{F}$-ring of fractions $\mathcal{D}_{\mathfrak{F}}$, and consider a crossed product ring $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$. Then, the following hold:*

a) $\mathcal{S}$ is a pseudo-Sylvester domain if and only if every finitely generated projective $\mathcal{S}$-module is stably free.

b) $\mathcal{S}$ is a Sylvester domain if and only if it is projective-free.

In any of the previous situations, the crossed product $\mathfrak{F} * \mathbb{Z}$ can be extended to a crossed product $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ and $\mathcal{D}_{\mathcal{S}} = \mathcal{Q}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$, the Ore division ring of fractions of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, is the universal division $\mathcal{S}$-ring of fractions. Furthermore, it is isomorphic to the universal localization of $\mathcal{S}$ with respect to the set of all stably full (resp. full) matrices.

As a particular application of the previous theorem, we obtained the next result through the recent advances on the Farrell–Jones conjecture by Bestvina–Fujiwara–Wigglesworth in [BFW19] and Brück–Kielak–Wu in [BKW19].

Theorem (Theorem 3.5.13). *Let E be a division ring and G a group arising as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is a free group. Then any crossed product $E * G$ is a pseudo-Sylvester domain. In particular, $\mathcal{D}_{E*G} = \mathcal{Q}(\mathcal{D}_{E*F} * \mathbb{Z})$ is the universal division $E * G$-ring of fractions and it is isomorphic to the universal localization of $E * G$ with respect to the set of all stably full matrices. Moreover, $E * G$ is a Sylvester domain if and only if it has stably free cancellation.

Every crossed product $E * G$ of a division ring E and a group G as in the previous theorem was known to admit a Hughes-free division ring of fractions, and it has been recently shown ([Jai20B, Theorem 3.7]) that in this particular case the Hughes-free division ring is also universal. Thus, the previous theorem provides an independent proof of the existence of the universal division $E * G$-ring of fractions and identify the precise set of matrices becoming invertible over this division ring.

Based on the fact that the Hughes-free division ring of fractions is also universal for this family of groups, the main result in [Grä20] and the veracity of the strong Atiyah conjecture in this case (see [Lin93]), we can give explicit realizations of $\mathcal{D}_{E*G}$. Before stating the result, note that since every group in the previous family is locally indicable, we can define a Conradian left order $\leq$ on G. We denote by $E((G, \leq))$ the space of Malcev-Neumann series, i.e., the E-vector space consisting of formal power series on G with coefficients in E and well-ordered support with respect to $\leq$.

Theorem (Proposition 3.4.26, Corollary 3.5.14, Corollary 4.4.5). *In the situation of the previous theorem, the universal division $E * G$-ring of fractions can be realized as the Dubrovin division ring, i.e., the division closure of $E * G$ inside* $\mathrm{End}(E((G, \leq)))$, *where $\leq$ is a Conradian left order in G. If $E = K$ is a subfield of $\mathbb{C}$, the universal division $K[G]$-ring of fractions can also be realized as the Linnell division ring, namely, the division closure of $K[G]$ inside $\mathcal{U}(G)$.*

The last two chapters are based on [JL20]. In **Chapter 4** we introduce the necessary ingredients to state and prove the strong Atiyah conjecture for locally indicable groups, namely, the theory of *epic $*$-regular rings* presented in [Jai19], the basic properties of the algebra $\mathcal{U}(G)$ of unbounded operators on $\ell^2(G)$ affiliated to the group von Neumann algebra $\mathcal{N}(G)$, and the theory of *rational semirings* developed in [DHS04] and [Sán08]. The inductive method based on the notion of complexity built in the latter references, together with the fact that $\mathcal{U}(G)$ admits a faithful Sylvester matrix rank function rk_G satisfying the analog of the Hughes-free condition for a division ring, allow us to prove the strong Atiyah conjecture for this family of groups and to identify the division closure of the group ring inside $\mathcal{U}(G)$ as the Hughes-free division ring of fractions. The main results of this chapter are then the following.

Theorem (Theorem 4.4.2, Corollary 4.4.3). *Let G be a countable locally indicable group and K a subfield of $\mathbb{C}$. Then G satisfies the strong Atiyah conjecture over K and the division closure $\mathcal{D}_{K[G]}$ of $K[G]$ in $\mathcal{U}(G)$ is the Hughes-free division $K[G]$-ring of fractions.*

Thanks to the suggestion of F. Henneke and D. Kielak, we also prove a stability result for the strong Atiyah conjecture in this context.

Proposition (Proposition 4.4.6). *Let K be a subfield of $\mathbb{C}$, G_2 a countable group arising as an extension*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

where G_1 is a torsion-free normal subgroup of G_2 and G_3 is locally indicable. If G_1 satisfies the strong Atiyah conjecture over K, then G_2 satisfies the strong Atiyah conjecture over K.

Finally, **Chapter 5** is devoted to explore the consequences of the previous results on the strong Atiyah conjecture and other related conjectures that can be tackled by means of the same techniques. In [Jai19], A. Jaikin-Zapirain posed three conjectures in relation to the aforementioned rank function rk_G on $\mathcal{U}(G)$ and the object which is at the core of the paper, namely, the $*$-regular closure $\mathcal{R}_{K[G]}$ of $K[G]$ in $\mathcal{U}(G)$ (for a subfield K of $\mathbb{C}$ closed under complex conjugation). All of the three conjectures were shown to hold in the same paper for sofic groups, and here we prove the corresponding statements for locally indicable groups.

The *independence conjecture* asks whether for a field K that can be embedded into $\mathbb{C}$ in different ways, and a matrix A over $K[G]$, the rk_G-rank of the image of A is independent of the embedding. This is the case for locally indicable groups.

Proposition (Proposition 5.1.1). *Let G be a countable locally indicable group, K a field of characteristic zero and $\varphi_1, \varphi_2 : K \to \mathbb{C}$ two different embeddings of K into $\mathbb{C}$. Then, for every matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$,*

$$\mathrm{rk}_G(\varphi_1(A)) = \mathrm{rk}_G(\varphi_2(A)).$$

The countability assumption can actually be dropped, since rk_G can still be defined for arbitrary groups and the proof goes analogously. With this and the main result of Chapter 4 we settle the existence of Hughes-free division rings of fractions for group rings over fields of characteristic zero.

Corollary (Corollary 5.1.2). *Let G be a locally indicable group and K a field of characteristic zero. Then there exists a Hughes-free division $K[G]$-ring of fractions.*

The *strong algebraic eigenvalue conjecture* asks about the algebraicity of the possible complex eigenvalues of a matrix A over $\mathcal{R}_{K[G]}$, or more generally, over the division closure $\mathcal{D}_{K[G]}$.

Proposition (Proposition 5.1.5). *Let G be a countable locally indicable group and K a subfield of $\mathbb{C}$. Then, for any $\lambda \in \mathbb{C}$ which is not algebraic over K and for any $A \in \mathrm{Mat}_n(\mathcal{D}_{K[G]})$, the matrix $A - \lambda I$ is invertible in $\mathcal{U}(G)$.*

The group von Neumann algebra $\mathcal{N}(G)$ of a countable ICC group is known to be a factor, meaning that $Z(\mathcal{N}(G)) = \mathbb{C}$, and this can be further extended to $\mathcal{U}(G)$, i.e. $Z(\mathcal{U}(G)) = \mathbb{C}$. The *center conjecture* asks whether the corresponding result holds when $\mathcal{N}(G)$ is substituted by the completion of $\mathcal{R}_{K[G]}$ with respect to the rk_G-metric described in the summary of Chapter 2. For a locally indicable group, $\mathcal{R}_{K[G]}$ is already rk_G-complete, since the strong Atiyah conjecture implies that $\mathcal{R}_{K[G]}$ is a division ring and

rk_G takes integer values on $\mathcal{R}_{K[G]}$. Moreover, we can prove the following slightly more general result replacing $\mathcal{R}_{K[G]}$ by the division closure $\mathcal{D}_{K[G]}$.

Proposition (Proposition 5.1.6, Corollary 5.1.7). *Let G be a countable locally indicable group, K a subfield of $\mathbb{C}$ and let $\mathcal{D}_{K[G]}$ denote the division closure of $K[G]$ in $\mathcal{U}(G)$. Then*

$$\mathcal{D}_{K[G]} \cap \mathbb{C} = K.$$

In particular, if G is ICC, $Z(\mathcal{D}_{K[G]}) = K$.

After discussing these three related conjectures, we turn our attention to *Lück's approximation conjecture in the space of marked groups.* One of the original forms of this approximation conjecture asked whether, for every connected compact CW-complex X with fundamental group G and for every nested chain of normal subgroups $\{G_i\}$ of G with $\bigcap G_i = \{1\}$, one can approximate the L^2-Betti numbers of the universal covering $\tilde{X}$ by means of the L^2-Betti numbers of the coverings $\bar{X}_i$ of X associated to the chain (cf. [Kam19, Conjecture 1.7]), i.e., for every $k \geq 0$,

$$\lim_{i\to\infty} b_k^{(2)}(\bar{X}_i) = b_k^{(2)}(\tilde{X}).$$

It was solved by W. Lück in [Lüc94] for the case in which G is residually finite, and since then various restatements and generalizations of the conjecture have been studied.

In particular, it can be phrased in terms of the rank function rk_G induced from $\mathcal{U}(G)$ on the group ring $K[G]$, and the rank functions associated to some "approximation" of G. For instance, in [Jai19] it was proved to hold when we consider sofic approximations of G. Here, we give a proof when we consider approximations in the space of marked groups and the group G being approximated is virtually locally indicable. More precisely, we prove the following.

Theorem (Theorem 5.2.13). *Let F be a finitely generated free group, let $\{M_i\}_{i\in\mathbb{N}}$ converge to M in the space of marked groups $\mathrm{MG}(F)$, set $G_i = F/M_i$, $G = F/M$, and let $\pi_G : \mathbb{C}[F] \to \mathbb{C}[G]$, $\pi_{G_i} : \mathbb{C}[F] \to \mathbb{C}[G_i]$ denote the induced homomorphisms. If G is virtually locally indicable then, for every matrix A over $\mathbb{C}[F]$,*

$$\lim_{i\to\infty} \mathrm{rk}_{G_i}(\pi_{G_i}(A)) = \mathrm{rk}_G(\pi_G(A)).$$

We finish the chapter by trying to apply the techniques from Chapter 4 to the question of universality of the Hughes-free division ring of fractions. We could not prove that, whenever it exists, the Hughes-free division $E * G$-ring of fractions is universal in the sense of P.M. Cohn, but we could prove the next proposition.

Proposition (Corollary 5.3.3). *Let $E*G$ be a crossed product of a division ring E and a locally indicable group G. If there exist a Hughes-free and a universal division $E*G$-ring of fractions, then they are isomorphic as $E*G$-rings.*

In particular, if G is countable, $E = K$ is a subfield of $\mathbb{C}$ and there exists a universal $K[G]$-ring of fractions, then it is isomorphic to the division closure of $K[G]$ in $\mathcal{U}(G)$.

Chapter 1

Rank functions

The notion of *rank* of a matrix is one of the most important concepts in linear algebra, and its relation to the study of solutions of a linear equation makes it probably the first and more natural approach to abstract algebra. Given a commutative field K and an $n \times m$ matrix A over K, there are several equivalent ways to define or characterize this concept:

a) As the number of linearly independent rows of A.

b) As the *dimension* of the image of the linear map $r_A : K^n \to K^m$ given by right multiplication by A.

c) As the number of linearly independent columns of A.

d) As the *dimension* of the image of the linear map $l_A : K^m \to K^n$ given by left multiplication by A.

e) As the size of the biggest invertible submatrix of A.

f) As the size of the biggest square submatrix of A with non-zero determinant.

g) Inductively, from a list of properties and methods to reduce the size of the matrix of interest.

From the equivalence of the previous definitions and the exactness of the associated dimension function $\dim_K$, one can deduce among many other properties, for instance, the following:

1. The rank of the $n \times n$ identity matrix I_n is n.

2. The rank of A equals the rank of its transpose A^T.

3. The rank of A is k if and only if there exists a $k \times k$ submatrix of A of maximum rank k.

4. The rank of A is k if and only if there exist an $n \times k$ matrix B and a $k \times m$ matrix C such that $A = BC$ and k is minimum with this property.

5. From the short exact sequence $0 \to \ker r_A \to K^n \to \operatorname{im} r_A \to 0$, we have that

$$\operatorname{rk}(A) = n - \dim_K(\ker r_A).$$

6. From the short exact sequence $0 \to \operatorname{im} r_A \to K^m \to \operatorname{coker} r_A \to 0$, we have that

$$\operatorname{rk}(A) = m - \dim_K(K^m/K^n A)$$

If one wants to develop a similar theory or to extend this notion over an arbitrary ring R, we observe here a range of possibilities and obstacles. Suppose that R is not commutative, so linear independence is not left-right symmetric. In view of a) and b), we may like to define the rank of A as the dimension of the left R-module spanned by the n rows of A, or we may want to use c) and d) instead to define it as the dimension of the right R-module generated by the m columns of A. These two quantities, usually referred to as the *row rank* and *column rank* of A, do not necessarily coincide, and hence lead to different extensions of the notion of rank.

Moreover, to define them we have relied on the existence of such a globally-defined notion of dimension. But unlike for commutative fields, not every R-module is free (i.e., has a basis), and even in this latter situation the number of elements in a basis is not necessarily unique. This is the case of rings *without invariant basis number* (or *non-IBN* rings), for which there exists an isomorphism of R-modules $R^m \cong R^n$ for some $n \neq m$.

In addition, non-commutativity makes evident the difficulties to define a determinant or to carry over properties like 2. to the more general setting.

On the one hand, if we want to define a symmetric (non-depending on a choice of sides) concept that makes sense on an arbitrary ring, then property 4. could fit our purpose. The number k appearing there is called the *inner rank* of A, and it can be extended to any ring. Nevertheless, sometimes we want the rank function to behave as desired on certain situations, like for example respecting property 1., and for this to hold we may need to restrict out attention to a certain subfamily of rings, like IBN rings or, as it will happen in most cases, the family of *stably finite* rings.

On the other hand, if we want to define a notion of rank function that naturally comes together with an associated notion of dimension for R-modules, we need to take into account at least the following two things. In the first place, the scope of definition of the associated dimension. In this sense, property 6. points initially to the family of finitely presented left R-modules M, i.e., those who admit an exact sequence of the form

$$R^n \xrightarrow{r_A} R^m \to M \to 0,$$

so that $M \cong R^m/R^n A$. Secondly, from here we also see that the properties of definition of this rank function must keep the number in 6. independent of the presentation matrix of M and that, if the defining properties are symmetric, then we also have an associated dimension for finitely presented right R-modules. The resulting rank function is called *Sylvester matrix rank function* and the associated dimension is called *Sylvester module* *rank function*, and were introduced by P.Malcolmson in [Mal80] within the context of Cohn's classification theory of epic division rings.

We start this chapter by defining and recalling in Section 1.1 the basic properties of the inner rank, with particular focus in the case of a division ring to see the resemblance with the commutative case, and we introduce the stable rank together with the family of stably finite rings. We then continue with the notions of Sylvester matrix and module rank function in Section 1.2, and then to a related notion on von Neumann regular rings in Section 1.3. During these sections we also study how all of these notions are related. In Section 1.4 we deal with situations in which it is possible to extend a rank function, either from finitely presented modules to arbitrary modules, or from a subring to the ring, and finally we explore in more depth one of these extensions, namely, the natural transcendental extension, in Section 1.5.

1.1 Inner and stable rank over a stably finite ring

In this section we introduce the inner rank and the stable rank of a matrix. Let R be an arbitrary ring and let A be an $n \times m$ matrix over R.

Definition 1.1.1. The *inner rank* $\rho(A)$ is defined as the least k such that A admits a decomposition $A = B_{n\times k}C_{k\times m}$. We say that a square matrix A of size $n \times n$ is *full* if $\rho(A) = n$.

Observe from the definition that the inner rank of a matrix satisfies

(Inn1) For any non-zero $a \in R$, $\rho(a) = 1$, and $\rho(0) = 0$.

(Inn2) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$, for any matrices A and B which can be multiplied.

(Inn3) $\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \leq \rho(A) + \rho(B)$ for any matrices A and B.

(Inn4) $\rho\begin{pmatrix} A \\ B \end{pmatrix} \geq \max\{\rho(A), \rho(B)\}$ and $\rho\begin{pmatrix} A & C \end{pmatrix} \geq \max\{\rho(A), \rho(C)\}$ for any matrices A, B, C of appropriate sizes.

However, there are a few desirable or expected properties that are not achieved unless we restrict our attention to a certain family of rings. For instance, if R is a ring without IBN, then there are positive integers $n > m$ such that $R^n \cong R^m$. In terms of matrices, this means that we can decompose the identity matrix I_n of order n as a product AB where A has size $n \times m$ and B has size $m \times n$, and thus $\rho(I_n) < n$. To avoid this behaviour, we are going to stick to the family of stably finite rings.

Definition 1.1.2. A ring R is said to be *stably finite* (or *weakly finite*) if for any two $n \times n$-matrices A and B over R such that $AB = I_n$, we also have $BA = I_n$.

Remark 1.1.3. This notion can be reformulated in terms of modules by saying that if $R^n \oplus K \cong R^n$, then $K = 0$. Indeed, the projection onto the first summand $R^n \cong R^n \oplus K \to R^n$ is given by right multiplication by some $n \times n$ matrix B. Since R^n is free, this splits and there exists a homomorphism $R^n \to R^n$, similarly defined by some matrix A, such that $AB = I_n$. By stably finiteness, $BA = I_n$, so the projection is an isomorphism and hence $K = 0$.

Conversely, if $AB = I_n$ then, in particular, the homomorphism r_B given by right multiplication by B is surjective. Thus, the sequence $0 \to \ker(r_B) \to R^n \xrightarrow{r_B} R^n \to 0$ is exact, and splits because R^n is free. Therefore, we have that $R^n \cong R^n \oplus \ker(r_B)$, and by hypothesis, $\ker(r_B) = 0$. This means that B is invertible and that $BA = BABB^{-1} = BI_nB^{-1} = I_n$. □

From this perspective, observe that over a stably finite ring we have $\rho(I_n) = n$ for every n, since the decomposition $I_n = A_{n\times m}B_{m\times n}$ with $n > m$ leads to an isomorphism $R^m \cong R^m \oplus (R^{n-m} \oplus \ker(r_B))$ from where $n - m = 0$.

For example, every division ring $\mathcal{D}$ is stably finite, since every left (right) $\mathcal{D}$-module is free of unique rank. Also, if K is a (commutative) field of characteristic 0 and G is any group, or if K has positive characteristic and G is sofic, the group ring $K[G]$ is stably finite (cf. [Jai19S, Corollary 13.7]). Furthermore, any subring of a stably finite ring is again stably finite.

Due to this latter property, and taking into account that throughout this document we are mainly interested in studying embeddings of rings into division rings, it seems rather natural to consider only stably finite rings. This family can also be characterized in terms of the so-called stable rank, a strengthened form of the inner rank which will play an important role once we introduce pseudo-Sylvester domains and universal division rings of fractions.

Definition 1.1.4. The *stable rank* $\rho^*(A)$ of an $n \times m$ matrix A is given by

$$\rho^*(A) = \lim_{s\to\infty} [\rho(A \oplus I_s) - s],$$

whenever the limit exists, where $A \oplus I_s$ denotes the block diagonal matrix with blocks A and I_s. We say that a square matrix is *stably full* if it has maximum stable rank.

Observe that from the definition of the inner rank it follows that the sequence in the limit is always non-increasing and bounded above by $\rho(A)$. In particular, for an $n \times n$ matrix A we obtain that $\rho^*(A) \leq \rho(A) \leq n$ and that $\rho^*(A) = n$ if and only if the sequence is constantly n. Thus, A is stably full if and only if $\rho(A \oplus I_s) = n + s$ for every $s \geq 0$.

With respect to this notion, we have the following characterization of stably finite rings ([Coh06, Proposition 0.1.3]).

Proposition 1.1.5. *A non-zero ring R is stably finite if and only if for every non-zero matrix A, the stable rank exists and $\rho^*(A) > 0$.*

The next lemma summarize useful properties of the stable rank over stably finite rings.

Lemma 1.1.6. *Let R be a stably finite ring and let A be a matrix over R.*

(1) For every $k \geq 0$, $\rho^*(A \oplus I_k) = \rho^*(A) + k$.

(2) There exists $N \geq 0$ such that for every $l \geq N$, $\rho^*(A \oplus I_l) = \rho(A \oplus I_l)$.

(3) $0 \leq \rho^*(A) \leq \rho(A)$.

Proof. Since R is stably finite, we know that $\rho^*(A) = r \geq 0$. This means that there exists $N \geq 0$ such that for any $l \geq N$ we have $\rho(A \oplus I_l) - l = r$. Thus, for $k \geq 0$,

$$\rho^*(A \oplus I_k) = \lim_{s \to \infty} [\rho(A \oplus I_k \oplus I_s) - (s + k) + k] = r + k = \rho^*(A) + k.$$

From here, we also deduce that for $l \geq N$ one has

$$\rho(A \oplus I_l) = l + r = l + \rho^*(A) = \rho^*(A \oplus I_l).$$

The last statement has already been observed above. □

There are two important families of rings that are defined in terms of the inner and the stable rank, namely, the Sylvester and pseudo-Sylvester domains, respectively. Sylvester domains were introduced by W. Dicks and E. Sontag in [DS78], generalizing the family of free ideal rings (or firs) that we define below, while pseudo-Sylvester domains were introduced analogously by P.M. Cohn and A.H. Schofield in [CS82].

Definition 1.1.7. A non-zero ring $\mathfrak{F}$ is a *free ideal ring* (or fir) if every left and every right ideal is a free $\mathfrak{F}$-module of unique rank.

As a consequence, in a fir every submodule of a free module is again free (cf. [Coh06, Corollary 2.1.2] and note that every submodule of a free R-module of rank κ is $\max(|R|, \kappa)$-generated).

For instance, every division ring is a fir. Also, P.M. Cohn proved that group rings $K[F]$ where K is a field and F is a free group are firs (cf. [Lew69, Theorem 1]). More generally, crossed products $E * F$ for any division ring E and free group F are firs (cf. [Sán08, Theorem 4.22 (i)]).

The definition of Sylvester and pseudo-Sylvester domains come from the following property of firs regarding the inner rank ([Coh06, Proposition 5.5.1]).

Proposition 1.1.8. *Let $\mathfrak{F}$ be a fir and let A, B be matrices over $\mathfrak{F}$ of sizes $n \times m$ and $m \times k$, respectively. If $AB = 0$, then*

$$\rho(A) + \rho(B) \leq m.$$

This property is usually referred to as Sylvester's law of nullity, from where the names of the previous two families were coined. More precisely,

Definition 1.1.9. A non-zero ring R is a *(pseudo-)Sylvester domain* if R is stably finite and satisfies Sylvester's law of nullity with respect to the inner (resp. stable) rank, i.e., for every $A \in \mathrm{Mat}_{n\times m}(R)$ and $B \in \mathrm{Mat}_{m\times k}(R)$ such that $AB = 0$, we have

$$\rho^{(*)}(A) + \rho^{(*)}(B) \leq m.$$

Observe that Sylvester and pseudo-Sylvester domains are actually domains. Indeed, if a and b are elements in a (pseudo)-Sylvester domain such that $ab = 0$, then the law of nullity tells us that $\rho^{(*)}(a) + \rho^{(*)}(b) \leq 1$. Since the inner (resp. stable) rank is a non-negative integer, this implies that either $\rho^{(*)}(a) = 0$ or $\rho^{(*)}(b) = 0$, in which case $a = 0$ or $b = 0$, respectively (see Proposition 1.1.5 for the stable case).

Sometimes, Sylvester's law of nullity with respect to the inner (resp. stable) rank is defined in the following apparently stronger way: for every $A \in \mathrm{Mat}_{n\times m}(R)$ and $B \in \mathrm{Mat}_{m\times k}(R)$,

$$\rho^{(*)}(AB) \geq \rho^{(*)}(A) + \rho^{(*)}(B) - m.$$

Nevertheless, this "stronger" form is actually equivalent to the previous one.

Lemma 1.1.10. *Let R be a (pseudo)-Sylvester domain. Then, for every $A \in \mathrm{Mat}_{n\times m}(R)$ and $B \in \mathrm{Mat}_{m\times k}(R)$, we have that*

$$\rho^{(*)}(AB) \geq \rho^{(*)}(A) + \rho^{(*)}(B) - m.$$

Proof. The result for the inner rank in a Sylvester domain is proved in [DS78] and in [Coh06, Corollary 5.5.2]. For the stable rank over a pseudo-Sylvester domain it is proved in [CS82, Section 6], and we add a proof for the sake of completeness.

Observe first that from the definition of the stable rank and from (Inn4),

$$\rho^*\begin{pmatrix} A \\ A' \end{pmatrix} = \lim_{s\to\infty}\left[\rho^*\begin{pmatrix} A & 0 \\ A' & 0 \\ 0 & I_s \end{pmatrix} - s\right] \geq \lim_{s\to\infty}\left[\rho^*\begin{pmatrix} A' & 0 \\ 0 & I_s \end{pmatrix} - s\right] = \rho^*(A'),$$

for any matrices A, A' of appropriate sizes. Similarly, $\rho^*\begin{pmatrix} A & A' \end{pmatrix} \geq \rho^*(A')$.

Let $A \in \mathrm{Mat}_{n\times m}(R)$ and $B \in \mathrm{Mat}_{m\times k}(R)$ be such that $\rho^*(AB) = l$. Invoking Lemma 1.1.6(1) and (2), there exists $s \geq 0$ such that $\rho(AB \oplus I_s) = l + s$. In particular, there exist an $(n+s)\times(l+s)$ matrix P and an $(l+s)\times(k+s)$ matrix Q such that $AB \oplus I_s = PQ$. Thus,

$$\begin{pmatrix} P & A\oplus I_s \end{pmatrix}\begin{pmatrix} -Q \\ B\oplus I_s \end{pmatrix} = -PQ + AB\oplus I_s = 0$$

and hence $\rho^*\begin{pmatrix} P & A\oplus I_s \end{pmatrix} + \rho^*\begin{pmatrix} -Q \\ B\oplus I_s \end{pmatrix} \leq l + m + 2s$. By the previous observation and using Lemma 1.1.6(1), we finally deduce that

$$\begin{aligned} \rho^*(AB) + m + 2s &\geq \rho^*\begin{pmatrix} P & A\oplus I_s \end{pmatrix} + \rho^*\begin{pmatrix} -Q \\ B\oplus I_s \end{pmatrix} \\ &\geq \rho^*(A\oplus I_s) + \rho^*(B\oplus I_s) = \rho^*(A) + \rho^*(B) + 2s, \end{aligned}$$

from where the result follows. □

Over a Sylvester domain, in particular over a division ring, the inner rank satisfies the following properties:

Lemma 1.1.11. *Let R be a Sylvester domain. Then the following hold for the inner rank:*

1. $\rho(1) = 1$ *and* $\rho(A) = 0$ *for any zero matrix A.*
2. $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$, *for any matrices A and B which can be multiplied.*
3. $\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \rho(A) + \rho(B)$ *for any matrices A and B.*
4. $\rho\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq \rho(A) + \rho(B)$ *for any matrices A, B, C of appropriate sizes.*

Proof. Properties 1. and 2. follow from the definition and were observed in (Inn1) and (Inn2). Now consider a matrix as in 4. where A, B, C have sizes $n \times m$, $r \times s$ and $n \times s$, respectively, and assume that it has inner rank k. Then, there exist matrices P, P', Q, Q' such that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} P_{n\times k} \\ P'_{r\times k} \end{pmatrix} \begin{pmatrix} Q_{k\times m} & Q'_{k\times s} \end{pmatrix}$$

Thus, $P'Q = 0$, from where the Sylvester's law of nullity tells us that $\rho(P') + \rho(Q) \leq k$. Therefore,

$$\begin{aligned} \rho\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} &\geq \rho(P') + \rho(Q) \overset{\text{(Inn2)}}{\geq} \rho(P'Q') + \rho(PQ) \\ &= \rho(A) + \rho(B) \overset{\text{(Inn3)}}{\geq} \rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \end{aligned}$$

In particular, we have 4. and we deduce 3. by setting $C = 0$. □

Remark 1.1.12. This lemma shows that the stably finiteness condition for Sylvester domains is in fact redundant. Indeed, let A and B be $n \times n$ matrices over a Sylvester domain such that $AB = I_n$. From Lemma 1.1.11(1),(2) and (3) we obtain that $n = \rho(I_n) = \rho(AB) \leq \rho(A)$, and hence $\rho(A) = n$. Since $A(BA - I_n) = 0$, we deduce from the law of nullity that

$$\rho(A) + \rho(BA - I_n) \leq n,$$

and therefore $\rho(BA - I_n) = 0$. Thus, $BA = I_n$ as we wanted to show.

Sometimes, however, keeping stably finiteness as a condition allows us to state at once results for Sylvester and pseudo-Sylvester domains in a homogeneous form. □

In the upcoming section, we define the notion of Sylvester matrix rank function over an arbitrary ring as a map that assigns to any matrix a non-negative real number and that satisfies the previous properties. It is important to note from Lemma 1.1.6(1) and (2), and from Lemma 1.1.11(1) and (3), that over a Sylvester domain the inner and the stable rank coincide.

Corollary 1.1.13. *Let R be a Sylvester domain and let A be a matrix over R. Then $\rho(A) = \rho^*(A)$. In particular, a square matrix is full if and only if it is stably full.*

We revisit Sylvester and pseudo-Sylvester domains in Chapter 3, where we recall some of their homological properties and introduce a recognition principle to identify pseudo-Sylvester domains. This is part of a joint work with Fabian Henneke developed in [HL20].

We finish this section showing that the inner rank on a division ring $\mathcal{D}$ is just the usual $\mathcal{D}$-rank (compare with the situation over a commutative field). The proof written here is a slight variation of the one in [Lam03, Exercises 13.13 & 13.14].

Proposition 1.1.14. *Let $\mathcal{D}$ be a division ring and let A be an $n \times m$ matrix over $\mathcal{D}$. The following quantities are equal:*

1. *The dimension of the left $\mathcal{D}$-module generated by the n rows of A, i.e., the row rank $\rho_r(A)$ of A.*
2. *The dimension of the right $\mathcal{D}$-module generated by the m columns of A, i.e., the column rank $\rho_c(A)$ of A.*
3. *The inner rank $\rho(A)$ of A.*
4. *The size of the biggest invertible (square) submatrix of A.*

The common value will be denoted $\mathrm{rk}_{\mathcal{D}}(A)$.

Proof. Assume first that $\rho_r(A) = k$, and let C be the $k \times m$ matrix consisting of the k left $\mathcal{D}$-linearly independent rows of A. Since the other rows are left $\mathcal{D}$-linear combinations of the rows of C, there exists an $n \times k$ matrix B such that $A = BC$, and thus $\rho(A) \leq \rho_r(A)$. Conversely, if $A = BC$ for some $n \times k$ and $k \times m$ matrices B and C, then any left linear dependence between the rows of B gives rise to a left linear dependence between the rows of A, so $\rho_r(A) \leq \rho_r(B)$. But the n rows of B generate a subspace of $\mathcal{D}^k$, from where necessarily $\rho_r(B) \leq k$, i.e., $\rho_r(A) \leq \rho(A)$. Therefore, $\rho(A) = \rho_r(A)$.

A symmetric argument using columns shows that $\rho(A) = \rho_c(A)$ and we have the equivalence between the first three statements of the proposition.

To see the equivalence with the last statement, consider first the case in which A is $n \times n$ of rank n. Then, the rows of A generate $\mathcal{D}^n$, so we can write the elements of the canonical basis of $\mathcal{D}^n$ as left $\mathcal{D}$-linear combinations of these rows, i.e., there exists an $n \times n$ matrix B such that $BA = I_n$. Since a division ring is stably finite, this means that A is invertible. Conversely, if A is invertible with inverse B, then $n = \rho_r(I_n) = \rho_r(AB) \leq \rho_r(A) \leq n$.

For the general case, note that if A has rank k then the submatrix C consisting of the k left $\mathcal{D}$-linearly independent rows of A has rank k. Since row and column rank coincide, there are k right $\mathcal{D}$-linearly independent columns in C, which form a $k \times k$ submatrix of A of rank k, and hence it is invertible. On the other hand, observe from the equality between row and column rank (or from Lemma 1.1.11 1., 2. and 3.), that the rank is invariant under multiplication by invertible matrices. Hence, if A has a $k \times k$ invertible submatrix A_1, we can assume without loss of generality that A_1 consists of the first k

rows and columns and thus

$$\begin{aligned}\rho(A) &= \rho\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \rho\left(\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} A_1^{-1} & 0 \\ 0 & I_{m-k} \end{pmatrix}\right) \\ &= \rho\begin{pmatrix} I_k & A_2 \\ A_3' & A_4 \end{pmatrix} \overset{\text{(Inn4)}}{\geq} \rho(I_k) = k.\end{aligned}$$

Putting everything together, we have proved that the rank coincides with the maximum size of an invertible submatrix of A, what finishes the proof. $\square$

The equivalence between 1. and 3. can be proved in the more general setting of left Bézout domains (cf. [Coh06, Proposition 5.4.4]), while an analog of the equivalence between 3. and 4. holds under some closure assumptions on the set of full matrices (cf. [Coh06, Proposition 5.4.9]).

To set the difference with respect to the usual rank on a commutative field, we state the following without proof (cf. [Lam03, Exercises 13.15]).

Proposition 1.1.15. *Let $\mathcal{D}$ be a division ring. If, for every matrix A over $\mathcal{D}$, we have* $\mathrm{rk}_{\mathcal{D}}(A) = \mathrm{rk}_{\mathcal{D}}(A^T)$*, then $\mathcal{D}$ is commutative.*

1.2 Sylvester rank functions

We have seen in Proposition 1.1.14 and Lemma 1.1.11 some of the basic properties of the usual rank on a division ring $\mathcal{D}$. The notion of Sylvester matrix rank function, conceived by Malcolmson in [Mal80] under the name *algebraic rank function*, describes a map that satisfies the properties showed in Lemma 1.1.11.

Definition 1.2.1. A *Sylvester matrix rank function* rk on a ring R is a function that assigns a non-negative real number to each matrix over R and that satisfies the following conditions.

(SMat1) $\mathrm{rk}(A) = 0$ if A is any zero matrix and $\mathrm{rk}(1) = 1$;

(SMat2) $\mathrm{rk}(AB) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$ for any matrices A and B which can be multiplied;

(SMat3) $\mathrm{rk}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{rk}(A) + \mathrm{rk}(B)$ for any matrices A and B;

(SMat4) $\mathrm{rk}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq \mathrm{rk}(A) + \mathrm{rk}(B)$ for any matrices A, B and C of appropriate sizes.

The motivation to introduce them is related to Cohn's classification theory of epic division rings. In this sense, it is not only true that the usual rank of a division ring is an integer-valued Sylvester matrix rank function, but that any integer-valued Sylvester matrix rank function rk on a ring R comes from a division ring, meaning that there exists a division ring $\mathcal{D}$ and a ring homomorphism $\varphi : R \to \mathcal{D}$ such that $\mathrm{rk} = \mathrm{rk}_{\mathcal{D}} \circ \varphi$. We develop this further after introducing epic division rings in Chapter 3.

We list and prove some of the basic properties of Sylvester matrix rank functions. Most of them come from [Jai19, Proposition 5.1] and [Jai19S, Proposition 5.1], and we often omit referencing this list when using them.

Properties 1.2.2. *Let R be a ring and let* rk *be a Sylvester matrix rank function on R. For all matrices of appropriate sizes:*

(1.) *If A has size $n \times m$, then* $\operatorname{rk}(A) \leq \min\{n, m\}$.

(2.) *If A is invertible of size $n \times n$, then* $\operatorname{rk}(A) = n$.

(3.) $\operatorname{rk}(A + B) \leq \operatorname{rk}(A) + \operatorname{rk}(B)$.

(4.) $\operatorname{rk}\begin{pmatrix} A & 0 \end{pmatrix} = \operatorname{rk}\begin{pmatrix} A \\ 0 \end{pmatrix} = \operatorname{rk}(A)$ *for any zero matrix.*

(5.) *If $A \in \operatorname{Mat}_{n\times m}(R)$ and $B \in \operatorname{Mat}_{m\times k}(R)$, then*

$$\operatorname{rk}(AB) \geq \operatorname{rk}(A) + \operatorname{rk}(B) - m$$

.

(6.) *Multiplying by square matrices of maximum rank does not change the rank.*

Proof. (1.) From (SMat1) and (SMat3), we have that $\operatorname{rk}(I_n) = n$ for every n. Thus,

$$\operatorname{rk}(A) = \operatorname{rk}(AI_m) \overset{\text{(SMat2)}}{\leq} \operatorname{rk}(I_m) = m$$

and analogously, $\operatorname{rk}(A) \leq n$.
(2.) If A is invertible of size $n \times n$,

$$n = \operatorname{rk}(I_n) = \operatorname{rk}(AA^{-1}) \overset{\text{(SMat2)}}{\leq} \operatorname{rk}(A) \overset{(1.)}{\leq} n$$

(3.) Assume that A and B have both size $n \times m$. Then

$$\begin{aligned} \operatorname{rk}(A + B) &= \operatorname{rk}\left(\begin{pmatrix} I_n & I_n \end{pmatrix}\begin{pmatrix} A & 0_{n\times m} \\ 0_{n\times m} & B \end{pmatrix}\begin{pmatrix} I_m \\ I_m \end{pmatrix}\right) \\ &\overset{\text{(SMat2)}}{\leq} \operatorname{rk}\begin{pmatrix} A & 0_{n\times m} \\ 0_{n\times m} & B \end{pmatrix} \overset{\text{(SMat3)}}{=} \operatorname{rk}(A) + \operatorname{rk}(B) \end{aligned}$$

(4.) For instance, if A is an $n \times m$ matrix,

$$\begin{aligned} \operatorname{rk}(A) &= \operatorname{rk}\left(\begin{pmatrix} A & 0 \end{pmatrix}\begin{pmatrix} I_m \\ 0 \end{pmatrix}\right) \overset{\text{(SMat2)}}{\leq} \operatorname{rk}\begin{pmatrix} A & 0 \end{pmatrix} \\ &= \operatorname{rk}(A\begin{pmatrix} I_m & 0 \end{pmatrix}) \overset{\text{(SMat2)}}{\leq} \operatorname{rk}(A) \end{aligned}$$

and the other equality is proved similarly.

(5.) Indeed, we have

$$\begin{aligned}\operatorname{rk}(AB)+m &\overset{\text{(SMat3)}}{=} \operatorname{rk}\begin{pmatrix} AB & 0_{n\times m} \\ 0_{m\times k} & I_m \end{pmatrix} \\ &\overset{\text{(SMat2)}}{\geq} \operatorname{rk}\left(\begin{pmatrix} 0_{m\times n} & -I_m \\ I_n & A \end{pmatrix}\begin{pmatrix} AB & 0_{n\times m} \\ 0_{m\times k} & I_m \end{pmatrix}\begin{pmatrix} I_k & 0_{k\times m} \\ -B & I_m \end{pmatrix}\right) \\ &= \operatorname{rk}\begin{pmatrix} B & -I_m \\ 0_{n\times k} & A \end{pmatrix} \overset{\text{(SMat4)}}{\geq} \operatorname{rk}(A)+\operatorname{rk}(B),\end{aligned}$$

from where $\operatorname{rk}(AB) \geq \operatorname{rk}(A)+\operatorname{rk}(B)-m$.

(6.) If $A \in \operatorname{Mat}_{n\times m}(R)$, and $B \in \operatorname{Mat}_{m\times m}(R)$ satisfies $\operatorname{rk}(B)=m$, then from (5.) we obtain $\operatorname{rk}(AB) \geq \operatorname{rk}(A)$. Thus, equality follows from (SMat2). $\square$

Note from the previous properties that a ring R with a Sylvester matrix rank function rk must have IBN. Indeed if A and B are matrices of sizes $n \times m$ and $m \times n$, respectively, satisfying $I_n = AB$ and $BA = I_m$, then

$$n = \operatorname{rk}(I_n) = \operatorname{rk}(AB) \leq \operatorname{rk}(A) \leq m,$$

and similarly $m \leq n$, so $m = n$ and thus finitely generated free R-modules have unique rank.

Observe also that, given a homomorphism of free finitely generated left R-modules $\phi : F_1 \to F_2$ with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, respectively, we can find an $n \times m$ matrix A over R such that

$$\begin{bmatrix} \phi(v_1) \\ \vdots \\ \phi(v_n) \end{bmatrix} = A \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix},$$

so that the coordinates of $\phi(x)$ in terms of the basis $\{w_1, \ldots, w_m\}$ of F_2 are obtained from the coordinates of x in terms of the basis $\{v_1, \ldots, v_n\}$ by right multiplication by A. When we work with R^n and R^m, consisting of tuples of elements of R written as rows, we commonly fix the canonical bases so that $\phi = r_A$ is given by right multiplication by A. Any other choice of R-basis is obtained from the previous one through multiplication by invertible matrices, what leaves invariant each of the notions of rank introduced so far, and hence we can write $\operatorname{rk}(\phi)$ to mean the rank of any of its associated matrices.

Observe now the relation between Sylvester matrix rank functions, inner rank and Sylvester domains.

Corollary 1.2.3. *Let R be a non-zero ring. Then R is a Sylvester domain if and only if the inner rank is a Sylvester matrix rank function.*

Proof. On the one hand, we saw in Lemma 1.1.11 that the inner rank over a Sylvester domain is a Sylvester matrix rank function. On the other hand, if ρ is a Sylvester matrix rank function, then it satisfies Sylvester's law of nullity because of (SMat1) and Properties 1.2.2(5.), and we discussed in Remark 1.1.12 that this already implies that R is stably finite.

$\square$

We also need to state the following rather technical two properties, that will prove useful in the next chapter when we start studying and describing the space of Sylvester matrix rank functions for certain families of rings.

Lemma 1.2.4. *Let* rk *be a Sylvester matrix rank function on a ring R. Take an element $a \in R$ and set $b_i = \mathrm{rk}(a^i) - \mathrm{rk}(a^{i+1})$. Then, for every $i \geq 0$, $b_i \geq b_{i+1}$. In particular, if a is nilpotent and $a^{n+2} = 0$, then $\mathrm{rk}(a^n) \geq 2\,\mathrm{rk}(a^{n+1})$.*

Proof. From (SMat3), (SMat4) and Properties 1.2.2(2.) and (6.), we obtain:

$$\begin{aligned}
\mathrm{rk}(a^{n+2}) + \mathrm{rk}(a^n) &= \mathrm{rk}\begin{pmatrix} a^{n+2} & 0 \\ 0 & a^n \end{pmatrix} \\
&= \mathrm{rk}\left(\begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{n+2} & 0 \\ 0 & a^n \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}\right) \\
&= \mathrm{rk}\begin{pmatrix} a^{n+1} & 0 \\ a^n & a^{n+1} \end{pmatrix} \geq 2\,\mathrm{rk}(a^{n+1}).
\end{aligned}$$

□

The second lemma studies additivity of the rank under certain conditions. In particular, when A and B are orthogonal and idempotent, and when A and B are matrices over a cartesian product $R_1 \times R_2$ such that $A \in \mathrm{Mat}_{n\times m}(R_1 \times \{0\})$ and $B \in \mathrm{Mat}_{n\times m}(\{0\} \times R_2)$.

Lemma 1.2.5. *Let* rk *be a Sylvester matrix rank function on a ring R. Let $A, B \in \mathrm{Mat}_{n\times m}(R)$, and assume that there exist matrices $C \in \mathrm{Mat}_{n\times n}(R)$, $D \in \mathrm{Mat}_{m\times m}(R)$ such that $CA = A$, $BD = B$ and $AD = CB = 0$. Then $\mathrm{rk}(A + B) = \mathrm{rk}(A) + \mathrm{rk}(B)$.*

Proof. Since the rank is invariant under multiplication by invertible matrices, we have the following.

$$\begin{aligned}
\mathrm{rk}(A) + \mathrm{rk}(B) &= \mathrm{rk}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\
&= \mathrm{rk}\left(\begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_m & 0 \\ I_m & I_m \end{pmatrix}\right) \\
&= \mathrm{rk}\left(\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & 0 \\ A + B & B \end{pmatrix}\begin{pmatrix} I_m & -D \\ 0 & I_m \end{pmatrix}\right) \\
&= \mathrm{rk}\begin{pmatrix} A + B & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{rk}(A + B).
\end{aligned}$$

□

In the case of a division ring $\mathcal{D}$, the usual rank function $\mathrm{rk}_{\mathcal{D}}$ can be defined in terms of the associated dimension function $\dim_{\mathcal{D}}$, that associates to each left $\mathcal{D}$-module its rank as a free module. Similarly, to any Sylvester matrix rank function we can associate a notion of dimension, called Sylvester module rank function (originally, just *dimension function*).

In general, the Sylvester module rank function that we can associate to a matrix rank function is not going to be defined over all modules, but on finitely presented modules, to which we can associate a presentation matrix.

Definition 1.2.6. A *Sylvester module rank function* dim on R is a map that assigns a non-negative real number to each finitely presented left R-module and that satisfies the following properties.

(SMod1) $\dim(0) = 0$, $\dim R = 1$;

(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;

(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$

Note that the last inequality of (SMod3) is actually redundant, since we can obtain the result from the exact sequence $M_2 \to M_3 \to 0 \to 0$ and (SMod1), but keeping it as a defining property may help to understand them better. The same happens with non-negativity, since it already follows from the exact sequence $M \to 0 \to 0 \to 0$, (SMod1) and (SMod3). Notice also that if $M_1 \cong M_2$, then we deduce from (SMod1) and (SMod3) that $\dim(M_1) = \dim(M_2)$, so dim is invariant under R-isomorphisms.

For every finitely presented left module M there exists a short exact sequence

$$R^n \xrightarrow{r_A} R^m \to M \to 0,$$

where r_A is given by right multiplication by the $n \times m$ matrix A, i.e., $M \cong R^m/R^nA$. Conversely, any such matrix gives rise to the finitely presented left module $M = R^m/R^nA$. Thus, a naive way to associate a Sylvester module rank function dim to a Sylvester matrix rank function rk is, mirroring what happens in the case of a field or a division ring, to define

$$\dim(M) = m - \operatorname{rk}(A).$$

This relation actually defines a bijection between Sylvester module and matrix rank functions. This fact was proved by P.Malcolmson in [Mal80, Theorem 4 and subsequent discussion] for the case of integer-valued rank functions. We rewrite here the proof for the sake of completeness and fill in the necessary gaps when considering real-valued rank functions. The key result to prove the well definition of the relation is the following strengthened form of Schanuel's lemma ([Mal80, Lemma 2]).

Lemma 1.2.7. *If* $0 \to K_1 \to P_1 \xrightarrow{\pi_1} M \to 0$ *and* $0 \to K_2 \to P_2 \to M \xrightarrow{\pi_2} 0$ *are exact sequences of left* R*-modules with* P_1 *and* P_2 *projective and* $K_i \subseteq P_i$*, then there is an automorphism* φ *of* $P_1 \oplus P_2$ *such that* $\varphi(K_1 \oplus P_2) = P_1 \oplus K_2$.

Proof. Since P_i is projective, the functor $\operatorname{Hom}_R(P_i, \square)$ is exact. In particular, the induced map $\operatorname{Hom}(P_1, P_2) \xrightarrow{\pi_2^*} \operatorname{Hom}(P_1, M)$ is surjective, and hence there exists $f : P_1 \to P_2$ such that $\pi_1 = \pi_2^*(f) = \pi_2 f$. Similarly, there exists $g : P_2 \to P_1$ such that $\pi_2 = \pi_1 g$, and hence the following diagram commutes

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & P_2 \arrow[r, "g"] \arrow[d, "\pi_2"'] & P_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & P_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\mathrm{id}_M"] & M \arrow[r, "\mathrm{id}_M"] & M. \end{tikzcd}$$

From the equality $\pi_1 gf = \pi_1$ we have $\pi_1(\mathrm{id}_{P_1} - gf) = 0$, and therefore $\mathrm{im}(\mathrm{id}_{P_1} - gf) \subseteq \ker \pi_1 = K_1$. Analogously, $\mathrm{im}(\mathrm{id}_{P_2} - fg) \subseteq \ker \pi_2 = K_2$. Moreover, for every $x \in K_1 = \ker \pi_1$, we have $\pi_2 f(x) = \pi_1(x) = 0$, and hence $f(K_1) \subseteq \ker \pi_2 = K_2$. Symmetrically, $g(K_2) \subseteq K_1$. With this information, one can check that the R-homomorphisms

$$\varphi : P_1 \oplus P_2 \to P_1 \oplus P_2 \quad \text{and} \quad \varphi' : P_1 \oplus P_2 \to P_1 \oplus P_2$$

defined, for every $(x, y) \in P_1 \oplus P_2$, as $\varphi(x, y) = (x + g(y), -f(x) + y - fg(y))$ and $\varphi'(x, y) = (x - gf(x) - g(y), f(x) + y)$, are mutual inverses.

Furthermore, if $x \in K_1$, then $\varphi(x, y) = (x + g(y), -f(x) + (\mathrm{id}_{P_2} - fg)(y)) \in P_1 \oplus K_2$, so $\varphi(K_1 \oplus P_2) \subseteq P_1 \oplus K_2$, and similarly, $\varphi'(P_1 \oplus K_2) \subseteq K_1 \oplus P_2$. Since φ' is the inverse of φ, we deduce that $\varphi(K_1 \oplus P_2) = P_1 \oplus K_2$. □

We are ready to define the bijective correspondence between Sylvester matrix and module rank functions (cf. [Mal80, Theorem 4]).

Proposition 1.2.8. *Let R be a ring. There exists a bijective correspondence between Sylvester matrix rank functions and Sylvester module rank functions, given by:*

(i) If rk *is a Sylvester matrix rank function on R, then we can define a Sylvester module rank function by assigning to any finitely presented left R-module with presentation $M = R^m/R^nA$ for some $A \in Mat_{n\times m}(R)$, the value*

$$\dim(M) := m - \mathrm{rk}(A).$$

This value does not depend on the given presentation.

(ii) If dim *is a Sylvester module rank function on R, then we can define a Sylvester matrix rank function by assigning to each $A \in Mat_{n\times m}(R)$, the value*

$$\mathrm{rk}(A) := m - \dim(R^m/R^nA).$$

We say in this case that rk *and* dim *are* associated.

Proof. (i) Let us first prove that the given value does not depend on the presentation matrix. As always, for any $n \times m$ matrix A, let r_A denote the homomorphism of free left R-modules $r_A : R^n \to R^m$ given by right multiplication by A. Now, assume that we have two presentations of M,

$$R^n \xrightarrow{r_A} R^m \to M \to 0 \qquad R^k \xrightarrow{r_B} R^l \to M \to 0$$

By Lemma 1.2.7, there exists an automorphism φ of $R^m \oplus R^l$ such that $\varphi(R^nA \oplus R^l) = R^m \oplus R^kB$. In particular, $\mathrm{rk}(\varphi) = m + l$. If $A_1 \oplus A_2$ denotes the block diagonal matrix with blocks A_1 and A_2, the following is exact with surjective maps

$$\begin{tikzcd}
 & & R^m \oplus R^k \arrow[d, "r_{I_m \oplus B}"] \arrow[dll, dotted, "g_2"] \\
R^n \oplus R^l \arrow[r, two heads, "r_{A \oplus I_l}"'] \arrow[urr, dotted, "g_1"] & R^nA \oplus R^l \arrow[r, two heads, "\varphi"'] & R^m \oplus R^kB
\end{tikzcd}$$

Therefore, the lifting property for free modules allows us to define the dotted maps g_1 and g_2 making the diagram commutative. Since $\mathrm{rk}(\varphi) = m + l$, using (SMat2) and (SMat3) we obtain from the existence of g_1 making the diagram commutative that $m + \mathrm{rk}(B) \le \mathrm{rk}(A) + l$, and from the existence of g_2 the reverse inequality. Thus,

$$m - \mathrm{rk}(A) = l - \mathrm{rk}(B)$$

and $\dim(M)$ is independent of the presentation.

Now, let us prove that dim is a Sylvester module rank function.

(SMod1) follows from (SMat1).

(SMod2): Note that if M and M' are finitely presented with presentation matrices A and B, then $A \oplus B$ is a presentation matrix for $M \oplus M'$, and hence we obtain the property (SMod2) from (SMat3).

(SMod3): We check this property in two steps.

1. Assume first that we have a surjective homomorphism $g : M' \to M$, and consider any presentation $M' \cong R^m/R^nA$ of M'. Since M is finitely presented, this induces a presentation of M of the form $M \cong R^m/R^kB$ with $R^nA \subseteq R^kB$. Using the lifting property of free modules, there exists a matrix C such that the following commutes

$$\begin{tikzcd}
R^n \arrow[r, two heads, "r_A"] \arrow[d, "r_C"'] & R^m \arrow[r, two heads] \arrow[d, "\mathrm{id}_{R^m}"] & M' \arrow[r, two heads] \arrow[d, "g"] & 0 \\
R^k \arrow[r, two heads, "r_B"] & R^m \arrow[r, two heads] & M \arrow[r, two heads] & 0.
\end{tikzcd}$$

Thus, $A = CB$ and as a consequence of (SMat2) we get $\dim(M') \ge \dim(M)$.

2. Suppose now that we have an exact sequence $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ and presentations

$$R^n \xrightarrow{r_A} R^m \xrightarrow{p_1} M_1 \to 0, \quad R^k \xrightarrow{r_B} R^l \xrightarrow{p_3} M_3 \to 0$$

of M_1 and M_3, respectively. Then M_2 can be $(m + l)$-generated by the image of the generators of M_1 defined through p_1 and some preimages of the generators of M_3 defined through p_3. We can construct a surjective homomorphism $\varphi : R^m \oplus R^l \to M_2$ and a homomorphism $\psi : R^n \oplus R^k \to R^m \oplus R^l$ such that the following commutes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & R^n \arrow[r, "\iota_1'"] \arrow[d, "r_A"'] & R^n \oplus R^k \arrow[r, "\pi_2'"] \arrow[d, "\psi"] & R^k \arrow[r] \arrow[d, "r_B"] & 0 \\
0 \arrow[r] & R^m \arrow[r, "\iota_1"] \arrow[d, "p_1"'] & R^m \oplus R^l \arrow[r, "\pi_2"] \arrow[d, "\varphi"] & R^l \arrow[r] \arrow[d, "p_3"] & 0 \\
& M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

and $\operatorname{im}\psi \subseteq \ker\varphi$, where $\iota_1, \iota_1', \pi_2, \pi_2'$ denote the natural embeddings and projections. In particular, $\operatorname{im}\psi$ is finitely generated and the R-module $M_2' := (R^m \oplus R^l)/\operatorname{im}\psi$, which is then finitely presented, admits a surjection $M_2' \to (R^m \oplus R^l)/\ker\varphi \cong M_2$. The first step of the proof tells us that $\dim(M_2') \geq \dim(M_2)$.

We can realize ψ as right multiplication by some $(n+k) \times (m+l)$ matrix D, and if we write

$$D = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with A_{11} of size $n \times m$ and A_{22} of size $k \times l$, then from the commutativity of the previous diagram we deduce that, for every $x \in R^n$, $y \in R^l$,

$$\begin{aligned}
yA_{22} &= \pi_2\psi(0,y) = r_B\pi_2'(0,y) = yB, \\
(xA_{11}, xA_{12}) &= \psi(x,0) = \psi\iota_1'(x) = \iota_1 r_A(x) = (xA, 0).
\end{aligned}$$

Therefore, $A_{11} = A$, $A_{22} = B$ and $A_{12} = 0$. Since the matrix $\begin{pmatrix} 0 & I_{k_1} \\ I_{k_2} & 0 \end{pmatrix}$ is invertible for every choice of k_i, we deduce from (SMat4) that

$$\begin{aligned}
\operatorname{rk}(D) &= \operatorname{rk}\begin{pmatrix} A & 0 \\ A_{21} & B \end{pmatrix} = \operatorname{rk}\left(\begin{pmatrix} 0 & I_k \\ I_n & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ A_{21} & B \end{pmatrix}\begin{pmatrix} 0 & I_m \\ I_l & 0 \end{pmatrix}\right) \\
&= \operatorname{rk}\begin{pmatrix} B & A_{21} \\ 0 & A \end{pmatrix} \geq \operatorname{rk}(A) + \operatorname{rk}(B)
\end{aligned}$$

As a consequence, $M_2' \cong (R^m \oplus R^l)/(R^n \oplus R^k)D$ satisfies

$$\dim(M_2') = m + l - \operatorname{rk}(D) \leq m + l - \operatorname{rk}(A) - \operatorname{rk}(B) = \dim(M_1) + \dim(M_3).$$

Adding everything up, we have (SMod3), what finishes the proof of (i).

(ii) Here, we just need to prove that rk is a Sylvester matrix rank function.

(SMat1) follows from (SMod1) and (SMod2).

(SMat2): Take $A \in \operatorname{Mat}_{n\times m}(R)$ and $B \in \operatorname{Mat}_{m\times k}(R)$. On the one hand, since $R^nAB \subseteq R^mB \subseteq R^k$ we have a short exact sequence of finitely presented left R-modules

$$R^k/R^nAB \to R^k/R^mB \to 0 \to 0,$$

from where we obtain $\dim(R^k/R^mB) \leq \dim(R^k/R^nAB)$ using (SMod1) and (SMod3). Consequently,

$$\mathrm{rk}(AB) = k - \dim(R^k/R^nAB) \leq k - \dim(R^k/R^mB) = \mathrm{rk}(B).$$

On the other hand, one can check that the sequence of finitely presented modules

$$R^m \xrightarrow{f} R^k \oplus R^m/R^nA \xrightarrow{g} R^k/R^nAB \to 0$$

given by $f(x) = (r_B(x), x + R^nA)$ and $g(y, z + R^nA) = -y + r_B(z) + R^nAB$ is exact, and hence from (SMod1),(SMod2) and (SMod3) we deduce that

$$k + \dim(R^m/R^nA) \leq m + \dim(R^k/R^nAB).$$

Thus, $\mathrm{rk}(AB) = k - \dim(R^k/R^nAB) \leq m - \dim(R^m/R^nA) = \mathrm{rk}(A)$.

(SMat3): Take $A \in \mathrm{Mat}_{n\times m}(R)$ and $B \in \mathrm{Mat}_{k\times l}(R)$. The homomorphism

$$\varphi : R^{m+l} \to R^m/R^nA \oplus R^l/R^kB$$

given by $\varphi(x, y) = (x + R^nA, y + R^kB)$, where x is an m-tuple and y is an l-tuple is surjective with kernel $R^{n+k}(A \oplus B)$. Since dim is invariant under isomorphisms, we deduce from (SMod2) that

$$\begin{aligned}\mathrm{rk}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} &= m + l - \dim(R^{m+l}/R^{n+k}(A \oplus B)) \\ &= m + l - \dim(R^m/R^nA) - \dim(R^l/R^kB) = \mathrm{rk}(A) + \mathrm{rk}(B).\end{aligned}$$

(SMat4): Take $A \in \mathrm{Mat}_{n\times m}(R)$, $B \in \mathrm{Mat}_{k\times l}(R)$, and $C \in \mathrm{Mat}_{n\times l}(R)$. In the first place, we have an R-isomorphism

$$R^{m+l}/R^{n+k}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \to R^{l+m}/R^{k+n}\begin{pmatrix} B & 0 \\ C & A \end{pmatrix}$$

given by commuting the first m coordinates with the last l coordinates. Accordingly, the matrices involved have the same rk-rank. Secondly, if we set $D = \begin{pmatrix} B & 0 \\ C & A \end{pmatrix}$, the sequence of finitely presented left R-modules

$$R^l/R^kB \xrightarrow{f} R^{l+m}/R^{k+n}D \xrightarrow{g} R^m/R^nA \to 0$$

where $f(x + R^kB) = (x, 0) + R^{k+n}D$ and $g((x, y) + R^{k+n}D) = y + R^nA$, is exact. As a consequence of (SMod3), we have then

$$\begin{aligned}\mathrm{rk}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} &= \mathrm{rk}(D) = l + m - \dim(R^{l+m}/R^{k+n}D) \\ &\geq l + m - \dim(R^l/R^kB) - \dim(R^m/R^nA) = \mathrm{rk}(A) + \mathrm{rk}(B).\end{aligned}$$

This finishes the proof of (ii). Finally, that the correspondence is bijective is clear from the definitions in (i) and (ii). □

Sometimes it is useful to switch between these two languages since, for example, there are results that can be stated in terms of modules but do not have an evident analog for matrices. Besides matrix and module rank functions, there are other variants of Sylvester rank functions, equivalent to the previous ones in the sense of Proposition 1.2.8, that shall not be discussed here. For instance, *Sylvester map rank functions*, introduced by A. Schofield (cf. [Sch85, Chapter 7]) and defined on maps between finitely generated projective modules, *bivariant Sylvester module rank functions*, recently introduced by H. Li, which are defined on the class of pairs of R-modules (M, M') such that $M \subseteq M'$ ([Li20, Definition 3.1 & Theorem 3.3]), and *extended Sylvester map rank functions*, introduced in the same paper and defined on all maps between R-modules ([Li20, Definition 6.1 & Theorem 6.2]).

1.3 Rank functions and von Neumann regular rings

In this section we work with von Neumann regular rings, a generalization of the notion of division ring, and we show that Sylvester matrix rank functions on such rings are determined by the values they take on elements. If we consider the usual rank $\mathrm{rk}_{\mathcal{D}}$ over a division ring $\mathcal{D}$ the previous claim is clear, since performing (left) row and (right) column operations (i.e., multiplying by invertible matrices) we can reduce every matrix A over $\mathcal{D}$ to a matrix with non-zero entries only in the main diagonal, and hence using the properties of $\mathrm{rk}_{\mathcal{D}}$ we see that $\mathrm{rk}_{\mathcal{D}}(A)$ is the number of non-zero entries in that diagonal. Moreover, although we will state this properly in the next chapter, this actually proves that $\mathrm{rk}_{\mathcal{D}}$ is the unique Sylvester matrix rank function on $\mathcal{D}$.

For this broader family of rings, we also relate the notions of Sylvester rank functions and *pseudo-rank functions* introduced by K. Goodearl (cf. [Goo91, Chapter 16]) as a generalization of the notion of rank function invented by J. von Neumann (cf. [vN98, Part II, Definition 18.1]). Most of the theoretical results of this section can be found precisely in [Goo91], which is an excellent almost self-contained reference to the study of von Neumann regular rings.

Definition 1.3.1. A ring $\mathcal{U}$ is *von Neumann regular* (or simply *regular*) if, for every $x \in \mathcal{U}$, there exists $y \in \mathcal{U}$ such that $xyx = x$.

As in the definition, we often reserve the letter $\mathcal{U}$ for von Neumann regular rings, since one of the main examples of regular rings considered in this document is the algebra of unbounded affiliated operators $\mathcal{U}(G)$ associated to a countable group G, which we introduce in Chapter 4. Also, unless otherwise specified we always use regular to mean von Neumann regular, so this should not be confused with the terminology of regular rings considered in commutative algebra.

Example 1.3.2. We list here some examples of von Neumann regular rings:

a) Division rings.

b) Direct products, direct limits and homomorphic images of regular rings. This follows from the definitions.

c) Matrix rings $\mathrm{Mat}_n(\mathcal{U})$ over regular rings $\mathcal{U}$ (cf. [Goo91, Theorem 1.7]).

d) Given a countable group G, the algebra of unbounded affiliated operators $\mathcal{U}(G)$ is regular (cf. [Lüc02, Theorem 8.22(3)]). As we already anticipated, this ring will appear later in Chapter 4.

□

There are several characterizations of von Neumann regular rings. The ones presented in the next proposition can be found in [Goo91, Theorem 1.1 & Corollary 1.13].

Proposition 1.3.3. *For a ring $\mathcal{U}$, the following statements are equivalent:*

(i) $\mathcal{U}$ is regular.

(ii) Every finitely generated left (right) ideal of $\mathcal{U}$ is generated by an idempotent.

(iii) Every left (right) $\mathcal{U}$-module is flat.

Recall here that, in general, a module P is finitely generated projective if and only if it is finitely presented flat (cf. [Rot09, Theorem 3.63]). Thus, we deduce the following from Proposition 1.3.3(iii).

Corollary 1.3.4. *If $\mathcal{U}$ is regular, then a left (right) $\mathcal{U}$-module is finitely presented if and only if it is finitely generated projective.*

Moreover, regular rings have a very particular structure of finitely generated projective modules. The properties in the next proposition come, respectively, from [Goo91, Theorem 1.11 & Proposition 2.6].

Proposition 1.3.5. *Let $\mathcal{U}$ be a regular ring and let P be a projective left $\mathcal{U}$-module. Then the following hold:*

(1.) Every finitely generated submodule of P is a direct summand of P, and hence projective. In particular, every finitely generated left ideal of $\mathcal{U}$ is projective.

(2.) If P is finitely generated, then P is a finite direct sum of cyclic submodules, each of which is isomorphic to a principal left ideal of $\mathcal{U}$.

The reason why we emphasize these properties is that, if we are given a Sylvester matrix rank function rk on a regular ring $\mathcal{U}$ with associated module rank function dim, then as a consequence of the previous properties we can deduce that dim is additive on short exact sequences and rk is uniquely determined by the values it takes on elements. Indeed, assume that we are given a short exact sequence of finitely presented left modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Since M_3 is projective by Corollary 1.3.4, there exists an isomorphism $M_2 \cong M_1 \oplus M_3$, and hence (SMod2) tells us that

$$\dim(M_1) + \dim(M_3) = \dim(M_2).$$

Moreover, if M is finitely presented, we deduce from Proposition 1.3.5 that there exists $x_1, \ldots, x_n \in \mathcal{U}$ such that $M \cong \mathcal{U}x_1 \oplus \cdots \oplus \mathcal{U}x_n$, and that each $\mathcal{U}x_i$ is projective. From the previous reasoning and the short exact sequences

$$0 \to \mathcal{U}x_i \to \mathcal{U} \to \mathcal{U}/\mathcal{U}x_i \to 0,$$

we deduce that

$$\mathrm{rk}(x_i) = 1 - \dim(\mathcal{U}/\mathcal{U}x_i) = 1 - (\dim(\mathcal{U}) - \dim(\mathcal{U}x_i)) = \dim(\mathcal{U}x_i)$$

and therefore

$$\dim(M) = \sum_{i=1}^{n} \dim(\mathcal{U}x_i) = \sum_{i=1}^{n} \mathrm{rk}(x_i).$$

Thus, two Sylvester matrix rank functions on a regular ring $\mathcal{U}$ that coincide on the elements of $\mathcal{U}$ are necessarily equal.

In [Goo91, Chapter 16], K. Goodearl introduces another notion of rank function on regular rings, a priori unrelated to Sylvester matrix rank functions and only defined on elements, and uses the above reasoning to associate a notion of dimension for finitely generated projectives. We will see later that this new notion extends uniquely to a Sylvester matrix rank function.

Definition 1.3.6. A *pseudo-rank function* on a regular ring $\mathcal{U}$ is a map $\mathrm{N} : \mathcal{U} \to [0, 1]$ such that

(PR1) $\mathrm{N}(1) = 1$;

(PR2) $\mathrm{N}(xy) \leq \min\{\mathrm{N}(x), \mathrm{N}(y)\}$ for all $x, y \in \mathcal{U}$;

(PR3) $\mathrm{N}(e + f) = \mathrm{N}(e) + \mathrm{N}(f)$ for all orthogonal idempotents $e, f \in \mathcal{U}$.

As usual, we deduce from (PR3) that $\mathrm{N}(0) = 0$. The corresponding dimension function for finitely generated projectives is the following:

Definition 1.3.7. A *(normalized) dimension function* on a regular ring $\mathcal{U}$ is a map d that assigns a real number to every finitely generated projective left $\mathcal{U}$-module and such that:

(D1) $\mathrm{d}(R) = 1$;

(D2) If $\mathrm{d}(P_1 \oplus P_2) = \mathrm{d}(P_1) + \mathrm{d}(P_2)$;

(D3) If $0 \to P_1 \to P_2$ is exact of finitely generated projectives, then

$$\mathrm{d}(P_1) \leq \mathrm{d}(P_2).$$

From (D2) we deduce that $\mathrm{d}(0) = 0$ and thus from (D3) that $\mathrm{d}(P) \geq 0$ for every P. Also from (D3), we see that d is invariant under isomorphism.

The association between pseudo-rank functions and dimension functions is described in the next proposition, corresponding to [Goo91, Proposition 16.8]).

Proposition 1.3.8. *Let $\mathcal{U}$ be a regular ring. There exists a bijective correspondence between pseudo-rank functions and dimension functions on $\mathcal{U}$, given by:*

(i) If d is a dimension function on $\mathcal{U}$, then we can define a pseudo-rank function N by assigning to each $x \in \mathcal{U}$ the value

$$\mathrm{N}(x) := \mathrm{d}(\mathcal{U}x)$$

(ii) If N is a pseudo-rank function on $\mathcal{U}$, then we can define a dimension function d on $\mathcal{U}$ by assigning to each finitely generated projective left $\mathcal{U}$-module P the value

$$\mathrm{d}(P) = \sum_{i=1}^{n} \mathrm{N}(x_i)$$

whenever $P \cong \mathcal{U}x_1 \oplus \cdots \oplus \mathcal{U}x_n$ for some $x_1, \ldots, x_n \in \mathcal{U}$.

With the previous association, we can show now the relation between Sylvester matrix and module rank functions and pseudo-rank and dimension functions.

Proposition 1.3.9. *Let $\mathcal{U}$ be a regular ring. Then every Sylvester module rank function is a dimension function, and viceversa. In particular, any pseudo-rank function extends uniquely to a Sylvester matrix rank function and any Sylvester matrix rank function arises in this way.*

Proof. We already discussed in Corollary 1.3.4 that Sylvester module rank functions and normalized dimension functions are defined on the same family of $\mathcal{U}$-modules.
Let d be a dimension rank function. Then:

(SMod1): $\mathrm{d}(R) = 1$ by (D1) and we already discussed after the definition of dimension function that $\mathrm{d}(0) = 0$.

(SMod2): This is precisely (D2).

(SMod3): Assume that we are given an exact sequence of finitely generated projectives $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3 \to 0$. Hence, we have short exact sequences

$$0 \to \ker g \to P_2 \to P_3 \to 0 \quad \text{and} \quad 0 \to \ker f \to P_1 \to \operatorname{im} f \to 0.$$

From the first one, we deduce that $P_2 \cong \ker g \oplus P_3$, and hence $\ker g$ is finitely generated and projective. From the second one, since $\operatorname{im} f = \ker g$ is finitely generated projective, we similarly deduce that $P_1 \cong \ker f \oplus \operatorname{im} f$ and that $\ker f$ is finitely generated projective. Therefore, since d is invariant under isomorphisms and non-negative, we deduce from (D3) that

$$\mathrm{d}(P_3) \leq \mathrm{d}(P_2) = \mathrm{d}(\ker g) + \mathrm{d}(P_3) = \mathrm{d}(\operatorname{im} f) + \mathrm{d}(P_3) \leq \mathrm{d}(P_1) + \mathrm{d}(P_3).$$

Therefore, d is a Sylvester module rank function.

Now, let dim be a Sylvester module rank function. Then:

(D1): This is part of (SMod1).

(D2): This is precisely the content of (SMod2).

(D3): Assume that we are given an exact sequence $0 \to P_1 \xrightarrow{f} P_2$ of finitely generated projectives. Since $\operatorname{im} f$ is a finitely generated submodule of P_2, Proposition 1.3.5(1.) tells us that there exists a submodule P_3 of P_2 such that $\operatorname{im} f \oplus P_3 = P_2$ and, in particular, P_3 is finitely generated projective. Since dim is invariant under isomorphisms, (SMod2) tells us that

$$\dim(P_1) = \dim(\operatorname{im} f) = \dim(P_2) - \dim(P_3) \leq \dim(P_2).$$

Therefore, dim is a dimension function.

To prove the last assertion of the proposition, let N be a pseudo-rank function on $\mathcal{U}$. By Proposition 1.3.8, it defines a unique dimension function d, which is then a Sylvester module rank function. By Proposition 1.2.8, d is associated to a unique Sylvester matrix rank function rk. We discussed after Proposition 1.3.5 that the rank is additive on short exact sequences, and therefore, for every element $x \in \mathcal{U}$,

$$\operatorname{rk}(x) = 1 - \mathrm{d}(\mathcal{U}/\mathcal{U}x) = \mathrm{d}(\mathcal{U}x) = \mathrm{N}(x).$$

Hence, rk is an extension of N. Conversely, if rk is a Sylvester matrix rank function, then we know that the rank of an element is non-negative and does not exceed 1, and that rk satisfies (PR1) and (PR2) by (SMat1) and (SMat2). Finally, if $e, f \in \mathcal{U}$ are orthogonal idempotents, then Lemma 1.2.5 for $A = C = e$ and $B = D = f$, tells us that $\operatorname{rk}(e + f) = \operatorname{rk}(e) + \operatorname{rk}(f)$, so rk satisfies (PR3) and its restriction to elements of $\mathcal{U}$ is a pseudo-rank function. □

In general, it is not easy to identify Sylvester matrix rank functions on a given ring, nor to prove that a given map satisfies (SMat1)-(SMat4). In this sense, the previous result allows us at least to restrict our attention to what happens at the level of elements when we work with regular rings.

Before passing to the next section, let us remark another property that is shared by certain Sylvester matrix rank functions on regular rings and the usual rank function on a division ring. For this, we need the following general definition.

Definition 1.3.10. Let R be a ring and let rk be a Sylvester matrix rank function on R. We define the *kernel of* rk, and we denote it by $\ker \operatorname{rk}$, as the set of elements in R whose rank equals zero, i.e.,

$$\ker \operatorname{rk} = \{r \in R : \operatorname{rk}(r) = 0\}$$

We say that rk is *faithful* if $\ker \operatorname{rk} = \{0\}$.

For instance, the usual rank $\operatorname{rk}_{\mathcal{D}}$ on a division ring is faithful, since every non-zero element has rank 1. We collect here some of the basic properties of $\ker \operatorname{rk}$.

Lemma 1.3.11. *Let R be a ring and let* rk *be a Sylvester matrix rank function on R. The following hold:*

a) $\ker \mathrm{rk}$ *is a two-sided ideal of* R.

b) rk *induces a faithful Sylvester matrix rank function on* $R/\ker \mathrm{rk}$.

c) rk *is faithful if and only if* $\mathrm{rk}(A) > 0$ *for every non-zero matrix* A.

Proof.
a) The kernel is not empty because $0 \in \ker \mathrm{rk}$. Take $s_1, s_2 \in \ker \mathrm{rk}$ and $r \in R$. By Properties 1.2.2(3.), $\mathrm{rk}(s_1 + s_2) \le \mathrm{rk}(s_1) + \mathrm{rk}(s_2) = 0$, and hence $s_1 + s_2 \in \ker \mathrm{rk}$. By (SMat2), $\mathrm{rk}(rs_1) \le \mathrm{rk}(s_1) = 0$ and $\mathrm{rk}(s_1 r) \le \mathrm{rk}(s_1) = 0$, and hence $rs_1, s_1 r \in \ker \mathrm{rk}$. Thus, $\ker \mathrm{rk}$ is a two-sided ideal.

b) Since $\ker \mathrm{rk}$ is a two-sided ideal of R, $R/\ker \mathrm{rk}$ is a ring. Consider the natural map $\pi : R \to R/\ker \mathrm{rk}$ and, for every matrix B over $R/\ker \mathrm{rk}$, pick a matrix A over R such that $\pi(A) = B$ and set $\mathrm{rk}'(B) = \mathrm{rk}(A)$.

To see that $\mathrm{rk}'(B)$ is well-defined, assume that A_1 and A_2 are matrices over R such that $\pi(A_1) = \pi(A_2) = B$. If we denote $C = A_1 - A_2$, and $C = (c_{ij})_{ij}$, then $\pi(C)$ is a zero matrix over $R/\ker \mathrm{rk}$, i.e., $c_{ij} \in \ker \mathrm{rk}$ for all i, j. Using repeatedly Properties 1.2.2(3.) and (4.) (together with (2.) and (6.)), we deduce that $\mathrm{rk}(C) \le \sum \mathrm{rk}(c_{ij}) = 0$. Thus, using again Properties 1.2.2(3.),

$$\mathrm{rk}(A_1) = \mathrm{rk}(C + A_2) \le \mathrm{rk}(C) + \mathrm{rk}(A_2) = \mathrm{rk}(A_2)$$

Similarly, $\mathrm{rk}(A_2) \le \mathrm{rk}(A_1)$, and hence we have equality. As a consequence, rk' defines a Sylvester matrix rank function on $R/\ker \mathrm{rk}$, which is faithful since $\mathrm{rk}'(a + \ker \mathrm{rk}) = 0$ if and only if $\mathrm{rk}(a) = 0$ if and only if $a \in \ker \mathrm{rk}$.

c) If $A = (a_{ij})_{ij}$ and E_{ij} denotes the matrix with 1 in position ij and zero everywhere else, then by (SMat1), (SMat2) and (SMat3),

$$\mathrm{rk}(A) \ge \mathrm{rk}(E_{1i} A E_{j1}) = \mathrm{rk}(a_{ij})$$

for all i, j. Hence, if rk is faithful and A is non-zero, then $\mathrm{rk}(A) > 0$. The other direction is clear. $\square$

The aforementioned property that regular rings share with division rings is that faithful Sylvester matrix rank functions identify invertible matrices.

Lemma 1.3.12. *Let* rk *be a faithful Sylvester matrix rank function on a regular ring* $\mathcal{U}$. *Then a square matrix* $A \in \mathrm{Mat}_n(\mathcal{U})$ *is invertible if and only if* $\mathrm{rk}(A) = n$.

Proof. We already proved that if A is invertible, then $\mathrm{rk}(A) = n$. Conversely, take $A \in \mathrm{Mat}_n(\mathcal{U})$ and assume that $\mathrm{rk}(A) = n$. By Example 1.3.2c), $\mathrm{Mat}_n(\mathcal{U})$ is regular and hence we can find $B \in \mathrm{Mat}_n(\mathcal{U})$ such that $ABA = A$. Therefore, using (SMat1) and Properties 1.2.2(6.),

$$0 = \mathrm{rk}(ABA - A) = \mathrm{rk}(A(BA - I_n)) = \mathrm{rk}(BA - I_n).$$

Since rk is faithful, we deduce from Lemma 1.3.11c) that necessarily $BA = I_n$. Similarly $AB = I_n$ and thus A is invertible. $\square$

1.4 Extending rank functions

In this section we deal with the problem of extending Sylvester rank functions in two different senses. On the one hand, the usual notion of dimension over a division ring $\mathcal{D}$ is defined for all $\mathcal{D}$-modules since every left or right $\mathcal{D}$-module is free of unique rank (of course, in this case the dimension is allowed to be $+\infty$), while for the moment Sylvester module rank functions are just defined on the family of finitely presented modules. Thus, it is natural to ask for conditions under which it is possible to extend Sylvester module rank functions to the family of all modules. On the other hand, assume that we are given a subring R of a ring S and a rank function on R. We are going to study some conditions under which it is possible to extend the rank from R to S. It will be of particular interest in this document the case in which $S = R[t;\tau]$, the skew polynomial ring with coefficients in R and with commutation rule $tx = \tau(x)t$, where τ is an automorphism of R.

Let R be a ring. Regarding the former sense of extension, we are going to study first the case in which it is not only possible to define a dimension for general R-modules but to guarantee a certain desirable behavior in resemblance to the case of division rings. Namely, the usual dimension on a division ring is additive on short exact sequences. If we want a similar behavior for a potential extension of a Sylvester module rank function, the following property looks rather natural.

Definition 1.4.1. A Sylvester module rank function dim on a ring R is *exact* if, for every surjection $\phi : M \twoheadrightarrow N$ of finitely presented left R-modules, we have

$$\dim(M) - \dim(N) = \inf\{\dim(L) : L \text{ finitely presented and } L \twoheadrightarrow \ker\phi\}.$$

Observe that in the previous setting $\ker\phi$ is just finitely generated. Notice for instance that, if we work in a regular ring, then M and N are projective, and hence we have an isomorphism $M \cong N \oplus \ker\phi$, from where $\ker\phi$ is finitely generated projective. In this case, we directly have by (SMod2),

$$\dim(M) - \dim(N) = \dim(\ker\phi).$$

Since any surjection $L \twoheadrightarrow \ker\phi$ implies $\dim(\ker\phi) \leq \dim(L)$ by (SMod3), we deduce the following.

Corollary 1.4.2. *Every Sylvester module rank function on a regular ring $\mathcal{U}$ is exact.*

The concept we are going to relate exact Sylvester module rank functions to is the concept of normalized length functions. Length functions were introduced by D.G. Northcott and M. Reufel in [NR65].

Definition 1.4.3. A *normalized length function* on a ring R is a map $\mathcal{L}$ defined on left R-modules and taking values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that:

(L1) $\mathcal{L}(0) = 0$, $\mathcal{L}(R) = 1$;

(L2) For every left R-module,

$$\mathcal{L}(M) = \sup\{\mathcal{L}(N) : N \text{ finitely generated and } N \leq M\};$$

(L3) For every exact sequence of left R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$,

$$\mathcal{L}(M_2) = \mathcal{L}(M_1) + \mathcal{L}(M_3).$$

It turns out that exact Sylvester module rank functions can be uniquely extended to normalized length functions, a result first proved by S. Virili in [Vir19B, Main Theorem] and later rediscovered through the notion of bivariant Sylvester module rank function by H. Li in [Li20, Corollary 4.3]. The procedure to extend the rank function is described in the following proposition.

Proposition 1.4.4. *Let* dim *be a Sylvester module rank function on a ring R. Then* dim *extends to a normalized length function if and only if it is exact. In this case, the extension is unique and is defined as follows. For every finitely generated left R-module N,*

$$\dim(N) = \inf\{\dim(L) : L \textit{ finitely presented and } L \twoheadrightarrow N\},$$

and for every left R-module M,

$$\dim(M) = \sup\{\dim(N) : N \textit{ finitely generated and } N \leq M\}.$$

Moreover, this yields a bijection between normalized length functions and exact Sylvester module rank functions.

In particular, every Sylvester module rank function on a regular ring can be extended to a normalized length function on R.

In general, if we drop the additivity requirement on short exact sequences, it is possible to extend any Sylvester module rank function to all left R-modules by means of the aforementioned bivariant Sylvester module rank functions. In this sense, the definition on finitely generated left R-modules would still be

$$\dim(N) = \inf\{\dim(L) : L \text{ finitely presented and } L \twoheadrightarrow N\},$$

while the expression for a general R-module M is

$$\dim(M) = \sup_{M_1} \inf_{M_2} \{\dim(M_2) - \dim(M_2/M_1)\},$$

where $M_1 \leq M_2$ are finitely generated submodules of M. When dim is exact, the extension is additive and hence the infimum equals $\dim(M_1)$, from where we recover the expression in Proposition 1.4.4.

Let us now turn our attention to the second sense of extension. Let R be a subring of a ring S, and observe that the restriction to R of any Sylvester matrix rank function on S is a Sylvester matrix rank function on R. More generally, for any ring homomorphism $\varphi : R \to S$ and any Sylvester matrix rank function rk_S on S, the map $\varphi^\sharp(\mathrm{rk}_S) := \mathrm{rk}_S \circ \varphi$ defines a Sylvester matrix rank function on R (where, for a matrix A over R, $\varphi(A)$ is taken entrywise). We may now wonder whether, given a Sylvester matrix rank function rk_R on R, we can find a Sylvester matrix rank function rk_S on S such that $\mathrm{rk}_R = \varphi^\sharp(\mathrm{rk}_S)$. When this is possible and S is regular, we say that rk_R is a regular rank.

Definition 1.4.5. A Sylvester matrix rank function rk on a ring R is *regular* if there exists a regular ring $\mathcal{U}$, a ring homomorphism $\varphi : R \to \mathcal{U}$ and a Sylvester matrix rank function rk' on $\mathcal{U}$ such that $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}') := \mathrm{rk}' \circ \varphi$. If, additionally, the rank rk' is faithful, then we say that $(\mathcal{U}, \varphi, \mathrm{rk}')$ is a *regular envelope of* rk.

Remark 1.4.6. Every regular Sylvester matrix rank function has a regular envelope. Indeed, let rk be a regular Sylvester matrix rank function on R and take any regular ring $\mathcal{U}$, homomorphism $\varphi : R \to \mathcal{U}$ and Sylvester matrix rank function rk' on $\mathcal{U}$ such that $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}')$. We proved in Lemma 1.3.11 that rk' induces a faithful Sylvester matrix rank function rk'' on $\mathcal{U}/\ker \mathrm{rk}$, which is also a regular ring. Moreover, if $\pi : \mathcal{U} \to \mathcal{U}/\ker \mathrm{rk}$ is the natural homomorphism, then from the proof of the lemma it follows that $\mathrm{rk}'(B) = \mathrm{rk}''(\pi(B))$ for every matrix B over $\mathcal{U}$. Therefore, for every matrix A over R,

$$\mathrm{rk}(A) = \mathrm{rk}'(\varphi(A)) = \mathrm{rk}''(\pi(\varphi(A))) = (\mathrm{rk}'' \circ \pi \circ \varphi)(A),$$

i.e., $(\mathcal{U}/\ker \mathrm{rk}', \pi \circ \varphi, \mathrm{rk}'')$ is a regular envelope of rk. □

We treat throughout the rest of this section three particular examples for which it is possible to extend the rank rk_R, while other examples will be given in the next chapter and after introducing universal localizations. We dedicate a subsection to each of them.

1.4.1 Extension to the Ore localization

The first example we deal with is the case in which S is the left (resp. right) Ore localization of R with respect to a multiplicative set T (i.e. a multiplicatively closed set containing 1_R) of non-zero-divisors in R satisfying the left (resp. right) Ore condition. The main definitions and its construction are briefly described in Section 3.1, and we denote S in this case by $T^{-1}R$.

The next proposition is a rewriting with full details of the proof presented in [Jai19S] for the final part of Proposition 5.2.

Proposition 1.4.7. *Let R be a ring, T a multiplicative set of non-zero-divisors of R satisfying the left Ore condition, and let* rk *be a Sylvester matrix rank function on R. Then* rk *extends to a Sylvester matrix rank function on $T^{-1}R$ if and only if*

$$\mathrm{rk}(t) = 1, \text{ for every } t \in T.$$

In this case, the extension is unique. Moreover, any Sylvester matrix rank function on $T^{-1}R$ is obtained in this way, and hence it is determined by its values on matrices over R.

Proof. The necessity of this condition relies on the fact that the elements in T become invertible in $T^{-1}R$, and therefore any rank on $T^{-1}R$ must assign them value 1. Moreover, if A is an $n \times m$ matrix over $T^{-1}R$, then reducing the entries of A to a common denominator we can write $A = t^{-1}A_1$, where $t \in T$, A_1 is a matrix over R (and $t^{-1}A_1$ stands for $(I_n t^{-1})A_1$). Since $I_n t^{-1}$ is an invertible matrix, then $\mathrm{rk}'(A) = \mathrm{rk}'(A_1)$ for every rank function rk' on $T^{-1}R$. Thus, rk' is determined by its values on matrices over R.

The previous discussion also indicates that there is a unique way to define rk for a matrix A over $T^{-1}R$: if $A = t^{-1}A_1$ for some matrix A_1 over R and $t \in T$, then we must set

$$\mathrm{rk}(A) = \mathrm{rk}(A_1)$$

Let us first check that this is well-defined. Let A be a matrix over $T^{-1}R$ and assume that we have two expressions $A = t_1^{-1}A_1 = t_2^{-1}A_2$, with $t_1, t_2 \in T$. The left Ore condition allows us to find $t \in T$ and $r \in R$ such that $tt_1 = rt_2$. In particular, since T is multiplicative, $tt_1 \in T$. Observe that from the equality

$$(tt_1)^{-1}tA_1 = t_1^{-1}A_1 = t_2^{-1}A_2 = (rt_2)^{-1}rA_2 = (tt_1)^{-1}rA_2$$

we obtain that $tA_1 = rA_2$ as matrices over R. Furthermore, by hypothesis $\mathrm{rk}(t) = \mathrm{rk}(tt_1) = \mathrm{rk}(t_2) = 1$ and thus, using Properties 1.2.2(6.), we deduce that $\mathrm{rk}(r) = 1$. From this, (SMat3) and again Properties 1.2.2(6.),

$$\mathrm{rk}(A_1) = \mathrm{rk}(tA_1) = \mathrm{rk}(rA_2) = \mathrm{rk}(A_2).$$

Therefore, the extension is well-defined. Note also that if we have an equality $t_1^{-1}A_1 = A_2t_2^{-1}$ with A_1, A_2 matrices over R and $t_1, t_2 \in T$, then we have the equality $t_1A_2 = A_2t_1$ over R and as a consequence, $\mathrm{rk}(A_1) = \mathrm{rk}(A_2)$. It remains to show that rk defines a Sylvester matrix rank function on $T^{-1}R$:

(SMat1): This is directly inherited from rk.

(SMat2): Let A,B be matrices over $T^{-1}R$ that can be multiplied, and write $A = t_1^{-1}A_1$, $B = s_1^{-1}B_1$ for some $t_1, s_1 \in T$, A_1, B_1 matrices over R. Since T is left Ore, we can find $t_2 \in T$ such that $A_1s_1^{-1} = t_2^{-1}A_2$ for some matrix A_2 over R, by first transforming right fractions into left fractions and then taking common denominator. As we already discussed, $\mathrm{rk}(A_1) = \mathrm{rk}(A_2)$, and hence

$$\begin{aligned}\mathrm{rk}(AB) &= \mathrm{rk}((t_2t_1)^{-1}A_2B_1) = \mathrm{rk}(A_2B_1) \leq \min\{\mathrm{rk}(A_2), \mathrm{rk}(B_1)\} \\ &= \min\{\mathrm{rk}(A_1), \mathrm{rk}(B_1)\} = \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}\end{aligned}$$

(SMat3): Let A, B be matrices over $T^{-1}R$, and write $A = t^{-1}A_1$, $B = t^{-1}B_1$ for a common denominator $t \in T$ and matrices A_1, B_1 over R. Then

$$\begin{aligned}\mathrm{rk}\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} &= \mathrm{rk}\left(t^{-1}\begin{pmatrix} A_1 & 0 \\ 0 & B_1\end{pmatrix}\right) = \mathrm{rk}\begin{pmatrix} A_1 & 0 \\ 0 & B_1\end{pmatrix} \\ &= \mathrm{rk}(A_1) + \mathrm{rk}(B_1) = \mathrm{rk}(A) + \mathrm{rk}(B)\end{aligned}$$

(SMat4): Similarly, let A,B,C be matrices over $T^{-1}R$ of appropriate sizes, and write $A = t^{-1}A_1$, $B = t^{-1}B_1$, $C = t^{-1}C_1$ for a common denominator $t \in T$ and matrices A_1, B_1, C_1 over R. Then

$$\begin{aligned}\mathrm{rk}\begin{pmatrix} A & C \\ 0 & B\end{pmatrix} &= \mathrm{rk}\left(t^{-1}\begin{pmatrix} A_1 & C_1 \\ 0 & B_1\end{pmatrix}\right) = \mathrm{rk}\begin{pmatrix} A_1 & C_1 \\ 0 & B_1\end{pmatrix} \\ &\geq \mathrm{rk}(A_1) + \mathrm{rk}(B_1) = \mathrm{rk}(A) + \mathrm{rk}(B)\end{aligned}$$

This finishes the proof. $\square$

1.4.2 The ultralimit construction

We already noticed before Definition 1.4.5 that, given a ring homomorphism $\varphi : R \to S$ and a Sylvester matrix rank function rk_S on S, we can define a Sylvester matrix rank function $\varphi^\sharp(\mathrm{rk}_S)$ on R by setting, for every matrix A over R, $\varphi^\sharp(\mathrm{rk}_S)(A) = \mathrm{rk}_S(\varphi(A))$. In particular, for any set X and for any family $\{R_i\}_{i\in X}$ of rings, we can define a Sylvester matrix rank function on $R = \prod_{i\in X} R_i$ from a Sylvester matrix rank function on any factor R_i through the canonical projection $R \to R_i$. In this subsection we show another way of constructing a Sylvester matrix rank function on $\prod_{i\in X} R_i$ when X is infinite (usually $X = \mathbb{N}$), this time starting with a rank function rk_i in each factor R_i. For this, we need to introduce ultrafilters and ultralimits of real numbers.

Definition 1.4.8. Let X be a set. A *filter on* X is a non-empty set ω of subsets of X satisfying:

1. $\varnothing \notin \omega$.
2. If $A \subseteq B \subseteq X$ and $A \in \omega$, then $B \in \omega$.
3. If $A, B \in \omega$, then $A \cap B \in \omega$.

An *ultrafilter on* X is a filter ω on X with the following additional property.

4. If $A \subseteq X$, then either $A \in \omega$ or $X\backslash A \in \omega$.

An ultrafilter ω on X is called *principal* if there exists $x \in X$ such that $\omega = \{A \subseteq X : x \in A\}$, in which case we denote $\omega = \omega_x$, and it is called *non-principal* otherwise.

Notice from the first and the third properties that the intersection of any two elements of a filter is non-empty. Thus, an ultrafilter contains one and only one among A and $X\backslash A$ for every $A \subseteq X$. Observe also the following properties of non-principal ultrafilters.

Lemma 1.4.9. *Let X be a set. The following hold:*

a) A non-principal ultrafilter on X cannot contain finite subsets of X.

b) There exists a non-principal ultrafilter on X if and only if X is infinite.

Proof. a) Let ω be a non-principal ultrafilter on X. Then ω cannot contain singletons. Indeed, assume that $\{x\} \in \omega$ for some $x \in X$. Since ω is closed under supersets, every subset of X containing x is in ω, and conversely, since the intersection of two elements in ω is a non-empty set in ω, we deduce that $x \in A$ for every $A \in \omega$, from where $\omega = \omega_x$.

Now, assume that $n \geq 2$ and that we have already showed that ω cannot contain subsets of X of cardinal $\leq n-1$. Let $A \subseteq X$ be a subset with n elements, and pick any $a \in A$. Since ω is an ultrafilter and both $\{a\}$ and $A' = A\backslash\{a\}$ are not in ω by the hypothesis of induction, then necessarily $X\backslash\{a\}, X\backslash A' \in \omega$. Thus, $X\backslash A = X\backslash\{a\} \cap X\backslash A' \in \omega$, and hence $A \notin \omega$.

b) We deduce from a) that if there exists a non-principal ultrafilter on X, then X must be infinite. Now assume that X is infinite. The set

$$\mathcal{F} = \{A \subseteq X : X \setminus A \text{ is finite}\}$$

is a filter on X, commonly known as the *Fréchet filter.* Consider the set

$$P = \{\omega : \omega \text{ filter on } X \text{ and } \mathcal{F} \subseteq \omega\}$$

partially ordered by inclusion. One can check that for every chain in P, the union of the elements in the chain is an upper bound in P, and thus, by Zorn's lemma there exists a maximal element $\omega_0 \in P$.

We claim that ω_0 is an ultrafilter. Otherwise, there exists $A_0 \subseteq X$ such that $A_0 \notin \omega_0$ and $X \setminus A_0 \notin \omega_0$. Note that, for every $A \in \omega_0$, $A \cap A_0 \neq \varnothing$, since otherwise we would have $A \subseteq X \setminus A_0$, and therefore $X \setminus A_0 \in \omega_0$, what contradicts our assumption. Now, the set ω' constructed from $\omega_0 \cup \{A_0\}$ by first adding all the intersections $A \cap A_0$ for $A \in \omega_0$ and then closing under supersets, is a filter on X properly containing ω_0 (in particular $\mathcal{F} \subseteq \omega'$), a contradiction.

Finally, ω_0 contains $\mathcal{F}$, and hence $X \setminus \{x\} \in \omega_0$ for all $x \in X$. Since ω_0 is an ultrafilter, then $\{x\} \notin \omega_0$ and consequently it cannot be principal. □

The notion of ultrafilter on a set X allows us to define the notion of ultralimit of a family of real numbers $\{a_x\}_{x \in X}$ in the following way.

Definition 1.4.10. Let X be a set, ω an ultrafilter on X and $\{a_x\}_{x \in X}$ a family of elements of $\mathbb{R}$. We say that $a \in \mathbb{R}$ is an *ultralimit* of $\{a_x\}_{x \in X}$ *with respect to* ω if for every $\epsilon > 0$, the set

$$\{x \in X : |a - a_x| < \epsilon\}$$

is an element of ω.

Although we shall not use them in this generality, ultralimits may be defined analogously on any topological space, and the properties needed for the existence and uniqueness of the ultralimit are compactness and Hausdorff. For further reading on the topic, one can consult, for instance, [DK18]. Here, we only prove the result for $\mathbb{R}$ to make this subsection self-contained (compare the proof with the one of [DK18, Lemma 10.25]).

Lemma 1.4.11. *Let X be a set, ω an ultrafilter on X and $\{a_x\}_{x \in X}$ a family of elements of $\mathbb{R}$. If $\{a_x\}_{x \in X}$ is bounded, then there exists a unique ultralimit $a \in \mathbb{R}$ with respect to ω, and we write $a = \lim_{\omega} a_i$.*

Proof. Let us start by proving the existence of an ultralimit. Since the family is bounded, we can choose a non-negative constant $C \in \mathbb{R}$ such $a_x \in [-C, C]$ for every $x \in X$. Assume, by contradiction, that $\{a_x\}_{x \in X}$ has no ultralimit. This means that, for every $a \in \mathbb{R}$, there exists $\epsilon_a > 0$ such that

$$A_{\epsilon_a} = \{x \in X : |a - a_x| < \epsilon_a\} \notin \omega$$

If $B_{\epsilon_a}(a)$ denotes the open ball with center a and radius ϵ_a, then we have the open cover $\bigcup_{a\in[-C,C]} B_{\epsilon_a}(a)$ of the compact set $[-C,C]$, and hence there exist $a_1,\dots,a_n \in [-C,C]$ such that $[-C,C] \subseteq \bigcup_{i=1}^n B_{\epsilon_{a_i}}(a_i)$.

Observe then that for every $x \in X$ there exists i such that $a_x \in B_{\epsilon_{a_i}}(a_i)$, and therefore $X = \bigcup_{i=1}^n A_{\epsilon_{a_i}}$ and $\varnothing = \bigcap_{i=1}^n X\setminus A_{\epsilon_{a_i}}$. Since $A_{\epsilon_{a_i}} \notin \omega$, the latter set is a finite intersection of elements in ω, and hence we deduce that $\varnothing \in \omega$, a contradiction. Thus, an ultralimit must exist for $\{a_x\}_{x\in X}$.

To prove uniqueness, assume that a and a' are two ultralimits of $\{a_x\}_{x\in X}$ with respect to ω, so that for every $\epsilon > 0$ we have that

$$A_\epsilon = \{x \in X : |a - a_x| < \epsilon\} \in \omega \quad \text{and} \quad A'_\epsilon = \{x \in X : |a' - a_x| < \epsilon\} \in \omega.$$

If $a \neq a'$ and we take $\epsilon = |a - a'|/3$, we would have $\varnothing = A_\epsilon \cap A'_\epsilon \in \omega$. Thus, we must have $a = a'$. □

Ultralimits share many properties with usual limits of sequences in $\mathbb{R}$. In the next lemma, we record three of them that we need later, and we also show why the most interesting ultrafilters are the non-principal ones.

Lemma 1.4.12. *Let X be a set, ω an ultrafilter on X and $\{a_x\}_{x\in X}$, $\{b_x\}_{x\in X}$ bounded families of real numbers with ultralimits a and b, respectively. Then*

a) $\lim\limits_{\omega} a_x + b_x = a + b$.

b) *If $a_x \leq b_x$ for every $x \in X$, then $a \leq b$.*

c) *If $a_x = c$ for every $x \in X$, then $a = c$.*

d) *If ω is principal, say $\omega = \omega_{x_0}$, then $a = a_{x_0}$.*

Proof. For every $\epsilon > 0$, define the sets

$$A_\epsilon = \{x \in X : |a - a_x| < \epsilon\} \quad \text{and} \quad B_\epsilon = \{x \in X : |b - b_x| < \epsilon\},$$

which are elements of ω, and set $C_\epsilon = A_\epsilon \cap B_\epsilon \in \omega$.

a) For every $\epsilon > 0$ and for every $x \in C_{\epsilon/2}$, we have

$$|a + b - a_x - b_x| \leq |a - a_x| + |b - b_x| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore, the set $\{x \in X : |(a + b) - (a_x + b_x)| < \epsilon\}$ is a superset of $C_{\epsilon/2}$ and hence an element of ω. This finishes the proof of *a)*.

b) Assume by contradiction that $a > b$ and take $\epsilon = (a - b)/2$. Then, for every $x \in C_\epsilon$, we have that $b_x < b + \epsilon = a - \epsilon < a_x$, and hence by the hypothesis we deduce that $\varnothing = C_\epsilon \in \omega$. Thus, necessarily $a \leq b$.

c) In this case, $\{x \in X : |c - a_x| < \epsilon\} = X$ for every $\epsilon > 0$. Observe that we always have $X \in \omega$ since ω is an ultrafilter and $\varnothing \notin \omega$. Thus, c is an ultralimit of $\{a_x\}_{x\in X}$, and by uniqueness, $a = c$.

d) For every $\epsilon > 0$, the set $\{x \in X : |a_{x_0} - a_x| < \epsilon\}$ contains x_0, and hence by definition it is in ω_{x_0}. Therefore, a_{x_0} is an ultralimit of $\{a_x\}_{x\in X}$, and by uniqueness, $a = a_{x_0}$. □

As we proved in Lemma 1.4.9, a non-principal ultrafilter cannot contain finite sets. Thus, somehow, a choice of ultrafilter is a choice of subsets of X that are going to be considered "large". When $X = \mathbb{N}$, this settles the relation between ultralimits and actual limits of sequences in $\mathbb{R}$.

Proposition 1.4.13. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a bounded sequence of real numbers. Then the limit $\lim\limits_{i\to\infty} a_i$ exists and equals a if and only if for every non-principal ultrafilter ω on $\mathbb{N}$, $a = \lim\limits_{\omega} a_i$.*

Proof. Assume first that the limit exists and equals a. Then, for every ϵ, the set $\{i \in \mathbb{N} : |a - a_i| \geq \epsilon\}$ is finite, and thus it cannot be contained in any non-principal ultrafilter. Therefore, for every non-principal ultrafilter ω on $\mathbb{N}$, its complement $\{i \in \mathbb{N} : |a - a_i| < \epsilon\}$ is an element of ω. Since this holds for every ϵ, we have that $\lim\limits_{\omega} a_i = a$.

Now, assume that a is not the limit of the sequence. Then, there exists $\epsilon_0 > 0$ such that the set $B = \{i \in \mathbb{N} : |a - a_i| \geq \epsilon_0\}$ is infinite. Consider the Fréchet filter $\mathcal{F}$ and the filter $\mathcal{F}' = \{A' \subseteq \mathbb{N} : B \subseteq A'\}$. If $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$, then, since A contains all but finitely many points of $\mathbb{N}$ and A' is infinite because it contains B, we deduce that $A \cap A' \neq \varnothing$. Thus, we can construct a filter ω from $\mathcal{F} \cup \mathcal{F}'$ by first adding all the intersections $A \cap A'$ for $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$ and then closing under supersets, and hence we can reason as in Lemma 1.4.9b) to see that there is a non-principal ultrafilter ω_0 containing ω. In particular, $B \subseteq \omega_0$ and hence its complement $\{i \in \mathbb{N} : |a - a_i| < \epsilon_0\}$ is not in ω_0, from where a is not the ultralimit of $\{a_i\}$ with respect to the non-principal ultrafilter ω_0. This finishes the proof. $\square$

The previous result is better understood when we relate ultralimits of $\{a_i\}$ with limits of convergent subsequences. More precisely, it can be proved that if a is the ultralimit of $\{a_n\}$ with respect to some non-principal ultrafilter ω on $\mathbb{N}$, then there exists a subsequence of $\{a_n\}$ converging to a, and conversely, if a subsequence of $\{a_n\}$ converges to a, then there exists a non-principal ultrafilter on $\mathbb{N}$ such that $a = \lim_\omega a_i$. In this sense, the previous proposition is just a reformulation of the fact that a sequence converges to a if and only if every convergent subsequence also converges to a.

We are now ready to define the ultralimit of a family of Sylvester matrix rank functions, with which we finish the subsection.

Proposition 1.4.14. *Let ω be an ultrafilter on a set X, and for every $x \in X$, let R_x be a ring together with a Sylvester matrix rank function rk_x. Consider the ring $R = \prod_{x\in X} R_x$ and let $\pi_x : R \to R_x$ be the natural projection. The expression*

$$\mathrm{rk}_\omega(A) = \lim_{\omega} \mathrm{rk}_x(\pi_x(A))$$

for every matrix A over R defines a Sylvester matrix rank function rk_ω on R. We call rk_ω the ultralimit of $\{\mathrm{rk}_x\}_{x\in X}$ on $\prod_{x\in X} R_x$ with respect to ω, and we write $\mathrm{rk}_\omega = \lim\limits_{\omega} \mathrm{rk}_x$.

Proof. Observe first that if A is an $n \times m$ matrix over R, then $\pi_x(A)$ is an $n \times m$ matrix over R_x, and therefore $\{\mathrm{rk}_x(\pi_x(A))\}_{x\in X}$ is a family of real numbers in $[0, \min\{n, m\}]$.

Hence, the ultralimit exists by Lemma 1.4.11 and it is non-negative by Lemma 1.4.12b) and c). Let us check that rk_ω is a Sylvester matrix rank function on R.

(SMat1): Since π_x is a ring homomorphism, $\mathrm{rk}_x(\pi_x(1_R)) = \mathrm{rk}_x(1_{R_x}) = 1$ for every $x \in X$. Thus, by Lemma 1.4.12c), $\mathrm{rk}_\omega(1) = 1$. Similarly $\mathrm{rk}_\omega(0) = 0$ for every zero matrix.

(SMat2): Again, since π_x is a ring homomorphism, given matrices A and B that can be multiplied, we have for every $x \in X$ that $\mathrm{rk}_x(\pi_x(AB)) = \mathrm{rk}_x(\pi_x(A)\pi_x(B)) \leq \mathrm{rk}_x(\pi_x(A))$, and therefore by Lemma 1.4.12b) we deduce that $\mathrm{rk}_\omega(AB) \leq \mathrm{rk}_\omega(A)$. Similarly $\mathrm{rk}_\omega(AB) \leq \mathrm{rk}_\omega(B)$.

(SMat3): If A and B are two matrices over R, then for every $x \in X$, $\mathrm{rk}_x(\pi_x(A \oplus B)) = \mathrm{rk}_x(\pi_x(A) \oplus \pi_x(B)) = \mathrm{rk}_x(\pi_x(A)) + \mathrm{rk}_x(\pi_x(B))$. From Lemma 1.4.12a), we deduce that $\mathrm{rk}_\omega(A \oplus B) = \mathrm{rk}_\omega(A) + \mathrm{rk}_\omega(B)$.

(SMat4:) Finally, let A, B, C be matrices of appropriate sizes. As above, we can check that for every $x \in X$,

$$\mathrm{rk}_x\left(\pi_x\left(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right)\right) \geq \mathrm{rk}_x(\pi_x(A) \oplus \pi_x(B)),$$

and hence by Lemma 1.4.12b), $\mathrm{rk}_\omega\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq \mathrm{rk}_\omega(A) + \mathrm{rk}_\omega(B)$.

This finishes the proof. □

We can also use the same strategy to define a new Sylvester rank function on a ring R from a family of Sylvester matrix rank functions on R. The resulting Sylvester matrix rank function is also called the ultralimit of the family. This should not cause any confusion since, as we see in the following proof, this can be seen as a particular case of the previous construction when each factor equals R.

Corollary 1.4.15. *Let R be a ring, let ω be an ultrafilter on a set X, and for every $x \in X$, let rk_x be a Sylvester matrix rank function on R. The expression*

$$\mathrm{rk}_\omega(A) = \lim_\omega \mathrm{rk}_x(A)$$

for every matrix A over R defines a Sylvester matrix rank function rk_ω on R. We call rk_ω the ultralimit of $\{\mathrm{rk}_x\}_{x\in X}$ on R and we write $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_x$.

Proof. The proof of Proposition 1.4.14 applies here. In fact, set $R_x = R$ for every $x \in X$ and consider $S = \prod_{x\in X} R_x$. Then we have a ring homomorphism $\iota : R \to S$ sending $r \to (r_x)_{x\in X}$ with $r_x = r$ for all x, and therefore we can pull back the ultralimit rk_ω of $\{\mathrm{rk}_x\}_{x\in X}$ on S to a Sylvester matrix rank function rk on R. This Sylvester matrix rank function is defined, for every matrix A over R, by

$$\mathrm{rk}(A) = \mathrm{rk}_\omega(\iota(A)) = \lim_\omega \mathrm{rk}_x(\pi_x(\iota(A))) = \lim_\omega \mathrm{rk}_x(A)$$

Thus rk is precisely what we have called the ultralimit of $\{\mathrm{rk}_x\}_{x\in X}$ on R. □

1.4.3 The natural transcendental extension

At the core of A. Jaikin-Zapirain's paper [Jai19] regarding the change of base field in both the strong Atiyah and Lück's approximation conjectures, there were two notions of natural extension of a rank function, namely, *the natural algebraic extension* and *the natural transcendental extension*, that appear when we consider an algebra over a field K and the corresponding extension of scalars associated to a field extension of K. A unifying treatment and a common generalization of both kinds of natural extension have been recently given in [JiLi21].

For our purposes, we introduce and further develop the notion of natural transcendental extension for the particular case of skew polynomial rings. More precisely, given an automorphism τ of R, we can try to extend a Sylvester matrix rank function rk on R to a Sylvester matrix rank function on $R[t;\tau]$ or $R[t^{\pm 1};\tau]$. Here, $R[t;\tau]$ denotes the *skew polynomial ring* with coefficients in R, which as an abelian group coincides with $R[t]$ but whose product is extended linearly from the commutation rule $tr = \tau(r)t$. On the other hand, $R[t^{\pm 1};\tau]$ is the *skew Laurent polynomial ring* with coefficients in R, which is defined analogously but allowing negative powers of t, and as we will recall in Example 3.1.7, coincides with the Ore localization of $R[t;\tau]$ with respect to the set of powers of t (cf. [Goo91, Exercise 10D]).

Observe that since τ is an automorphism, $R[t;\tau]$ is free both as a left and a right R-module with R-basis $\{t^i\}$. Given an element $p \in R[t;\tau]$, we can associate p with some map between free finitely generated R-modules. We have several ways to do this: again, since τ is an automorphism, the left ideal $R[t;\tau]t^n$ is actually two-sided and $R[t;\tau]t^n = t^n R[t;\tau]$ for every positive integer n. Thus, the quotient $R[t;\tau]/R[t;\tau]t^n$ is a ring which is free of rank n as a left R-module. We define then

$$\phi_n^p : R[t;\tau]/R[t;\tau]t^n \to R[t;\tau]/R[t;\tau]t^n$$

by $\phi_n^p(q + R[t;\tau]t^n) = qp + R[t;\tau]t^n$, which is an R-homomorphism of free left modules. We sometimes just say that ϕ_n^p is given by right multiplication by p because it coincides precisely with multiplication on the right by $p + R[t;\tau]t^n$ in the ring $R[t;\tau]/R[t;\tau]t^n$. Observe that we have constructed a map

$$\begin{array}{ccc} R[t;\tau] & \to & \mathrm{End}_R\left(R[t;\tau]/R[t;\tau]t^n\right) \\ p & \mapsto & \phi_n^p. \end{array}$$

If we let endomorphisms act on the right (or equivalently, if we work with the opposite endomorphism ring), the previous map is actually a ring homomorphism, and if we fix a basis of $R[t;\tau]/R[t;\tau]t^n$ the endomorphism ring is isomorphic to $\mathrm{Mat}_n(R)$. Therefore, we have (after fixing a basis), a ring homomorphism

$$\psi_n : R[t;\tau] \to \mathrm{Mat}_n(R) \tag{1.1}$$

in which $\psi_n(p)$ maps to the matrix associated to ϕ_n^p (with respect to this basis). In this latter ring, $\frac{1}{n}$ rk defines a Sylvester matrix rank function, and hence we have a Sylvester matrix rank function $\widetilde{\mathrm{rk}}_n = \psi_n^\sharp(\frac{1}{n}\,\mathrm{rk})$ on $R[t;\tau]$. In particular,

$$\widetilde{\mathrm{rk}}_n(p) = \frac{\mathrm{rk}(\psi_n(p))}{n} = \frac{\mathrm{rk}(\phi_n^p)}{n}.$$

It is useful to observe that $t^k p$ just twists the coefficients of p by τ^k and shifts them k positions to the right. Thus, if we build ψ_n from the canonical basis $\{1+R[t;\tau]t^n,\dots,t^{n-1}+R[t;\tau]t^n\}$ on ${}^{R[t;\tau]}\!\big/_{R[t;\tau]t^n}$, then for every $p = a_0 + a_1 t + \cdots + a_s t^s$ we have that

$$\psi_n(p) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & \\ 0 & \tau(a_0) & \tau(a_1) & \dots & \\ 0 & 0 & \tau^2(a_0) & \dots & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \dots & & \tau^{n-1}(a_0) \end{pmatrix} \tag{1.2}$$

By definition, for a $k\times l$ matrix A over $R[t;\tau]$, we define $\psi_n(A)$ entrywise, i.e., if $A=(a_{ij})$ then $\psi_n(A)$ is a $k\times l$ matrix over $\mathrm{Mat}_n(R)$ whose ij-entry is the matrix associated to $\phi_n^{a_{ij}}$ with respect to the fixed bases. Again, if we consider the canonical basis $\{1+R[t;\tau]t^n,\dots,t^{n-1}+R[t;\tau]t^n\}$, we can see that $\psi_n(A)$ coincides with the matrix associated to the map given by right multiplication by A,

$$\phi_n^A : \left({}^{R[t;\tau]}\!\big/_{R[t;\tau]t^n}\right)^k \to \left({}^{R[t;\tau]}\!\big/_{R[t;\tau]t^n}\right)^l$$

with respect to the same basis in each factor. Therefore,

$$\widetilde{\mathrm{rk}}_n(A) = \frac{\mathrm{rk}(\psi_n(A))}{n} = \frac{\mathrm{rk}(\phi_n^A)}{n}.$$

A couple of remarks are now in order:

1. Given the way we obtained the rank, we see that if $R=\mathcal{U}$ is regular, then $\widetilde{\mathrm{rk}}_n$ is a regular rank coming from $\mathrm{Mat}_n(\mathcal{U})$.

2. In general, $\widetilde{\mathrm{rk}}_n$ does not extend rk. If we consider an element $r\in R$, then the matrix associated to ϕ_n^r with respect to the canonical basis is

$$\begin{pmatrix} r & 0 & \dots & 0 \\ 0 & \tau(r) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \tau^{n-1}(r) \end{pmatrix}$$

   and hence $\widetilde{\mathrm{rk}}_n(r) = \frac{1}{n}\sum_i \mathrm{rk}(\tau^i(r))$. Similarly, if we take a matrix A over R, then the matrix associated to ϕ_n^A is equivalent to the block diagonal matrix with blocks $A,\tau(A),\dots,\tau^{n-1}(A)$. Therefore, we shall need some compatibility between rk and τ.

3. The rank $\widetilde{\mathrm{rk}}_n$ cannot be further extended to $R[t^{\pm 1};\tau]$. In the skew Laurent polynomial ring, the indeterminate t is invertible, and hence any rank function on $R[t^{\pm 1};\tau]$ should give t value 1. However, the matrix associated to ϕ_n^t with respect to the canonical basis is the matrix with ones over the main diagonal, and hence $\widetilde{\mathrm{rk}}_n(t) = \frac{n-1}{n}$.

To fix 2. we introduce the following definition.

Definition 1.4.16. Let R be a ring, τ an automorphism of R and rk a Sylvester matrix rank function on R. We say that rk is τ-*compatible* if for every matrix A over R, we have $\mathrm{rk}(A) = \mathrm{rk}(\tau(A))$.

We can rewrite this property in terms of the associated module rank function dim. Let M be a finitely presented left R-module, and denote by $t^k M$, $k \in \mathbb{Z}$, the finitely presented left R-module whose elements are of the form $t^k m$ for $m \in M$, with natural sum (i.e., it is isomorphic to M as an abelian group) but with R-product given by $r(t^k m) := t^k(\tau^{-k}(r)m)$.

Note here that if the presentation matrix of M is the $n\times m$ matrix A, then $R^m/R^n\tau^k(A) \cong t^k M$ via $v + R^n\tau^k(A) \mapsto t^k(\tau^{-k}(v) + R^n A)$. Observe also that the notation is consistent with $R[t^{\pm 1};\tau]$, in the sense that if M is an R-submodule of the $R[t^{\pm 1};\tau]$-module M', then $t^k M$ is an R-submodule of M' whose operations are precisely the ones defined above.

Lemma 1.4.17. *Let* rk *be a Sylvester matrix rank function on a ring R and* dim *its associated Sylvester module rank function. Let τ be an automorphism of R. Then* rk *is τ-compatible if and only if for every finitely presented R-module M, $\dim(M) = \dim(tM)$.*

Proof. As we mentioned earlier, for every matrix $A \in \mathrm{Mat}_{n\times m}(R)$, the finitely presented left R-modules $R^m/R^n\tau(A)$ and $t(R^m/R^n A)$ are isomorphic. Thus, if rk is τ-compatible, then

$$\begin{aligned}\dim(R^m/R^n A) &= m - \mathrm{rk}(A) = m - \mathrm{rk}(\tau(A)) \\ &= \dim(R^m/R^n\tau(A)) = \dim(t(R^m/R^n A)).\end{aligned}$$

Conversely, if $\dim(M) = \dim(tM)$ for every finitely presented R-module and we take a matrix $A \in \mathrm{Mat}_{n\times m}(R)$, then we can apply the same reasoning to the finitely presented module $R^m/R^n A$ to obtain that $\mathrm{rk}(\tau(A)) = \mathrm{rk}(A)$. □

Observe that if rk is τ-compatible associated to dim, then we have also $\mathrm{rk}(\tau^k(A)) = \mathrm{rk}(A)$ and $\dim(M) = \dim(t^k M)$ for every A, M and $k \in \mathbb{Z}$.

Definition 1.4.18. If τ is an automorphism of R and rk is a τ-compatible Sylvester matrix rank function on R, then the rank $\widetilde{\mathrm{rk}}_k$ defined above by

$$\widetilde{\mathrm{rk}}_k(A) = \frac{\mathrm{rk}(\psi_k(A))}{k} = \frac{\mathrm{rk}(\phi_k^A)}{k}$$

is called the k^{th} *extension of* rk to $R[t;\tau]$.

To fix 3., i.e., to be able to extend rk to $R[t^{\pm 1};\tau]$, instead of considering the k^{th} extensions we are going to consider their limit:

Remark 1.4.19. Let τ be an automorphism of R, rk a τ-compatible Sylvester matrix rank function on R and assume for the moment that the limit

$$\widetilde{\mathrm{rk}}(A) := \lim_{k\to\infty} \widetilde{\mathrm{rk}}_k(A)$$

exists for every A over $R[t;\tau]$. Then $\widetilde{\mathrm{rk}}$ defines a Sylvester matrix rank function on $R[t;\tau]$ that extends rk because each k^{th} extension enjoys these properties. Moreover,

$$\widetilde{\mathrm{rk}}(t) = \lim_{k\to\infty} \widetilde{\mathrm{rk}}_k(\phi_k^t) = \lim_{k\to\infty} \frac{k-1}{k} = 1.$$

Similarly, or directly using Properties 1.2.2(6.), $\widetilde{\mathrm{rk}}(t^n) = 1$ for every n. Thus, by Proposition 1.4.7, $\widetilde{\mathrm{rk}}$ can be uniquely extended to a rank function, denoted again by $\widetilde{\mathrm{rk}}$, on $R[t^{\pm 1};\tau]$.

The notation should not induce confusion: the same proposition describes how any Sylvester matrix rank function on $R[t^{\pm 1};\tau]$ is completely determined by the values on matrices over $R[t;\tau]$. Thus, if rk' is a Sylvester matrix rank function on $R[t^{\pm 1};\tau]$ whose restriction to $R[t;\tau]$ coincides with $\widetilde{\mathrm{rk}}$, then $\mathrm{rk}' = \widetilde{\mathrm{rk}}$ as a rank function on $R[t^{\pm 1};\tau]$. $\square$

At the time of writing [JL20], it was not known whether the extension $\widetilde{\mathrm{rk}}$ of a τ-compatible rank function rk on R always exists. This can now be shown as an application of the main results in [JiLi21]. As we shall recall later, $S = R[t^{\pm 1};\tau]$ is a particular instance of crossed product $S = R * \mathbb{Z} = \bigoplus_{i\in\mathbb{Z}} Rt^i$. Consider the set of finitely generated free left R-modules

$$\mathcal{F}(S) = \left\{ \sum_{i\in F} Rt^i : F \subseteq \mathbb{Z} \text{ finite} \right\}$$

and define, for an $n \times m$ matrix A over S and an element $\mathcal{W} \in \mathcal{F}(S)\setminus\{0\}$,

$$\mathrm{rk}_{\mathcal{W}}(A) = \mathrm{rk}(r_{A,\mathcal{W}} : \mathcal{W}^n \to \tilde{\mathcal{W}}^m)$$

where $\tilde{\mathcal{W}} \in \mathcal{F}(S)\setminus\{0\}$ is such that $\mathcal{W}a_{ij} \subseteq \tilde{\mathcal{W}}$ for every i, j and the map is given by right multiplication by A. The authors shown that $\mathrm{rk}_{\mathcal{W}}(A)$ does not depend on the choice of $\tilde{\mathcal{W}}$ ([JiLi21, Lemma 4.1]) and that in the previous setting (see [JiLi21, Examples 3.3, 5.3 & 7.3, Remark 5.6, Theorems 6.7 & 8.3], and observe that τ-compatibility is equivalent to their property of preservation of rk) one can define a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{F}}$ on S extending rk by

$$\mathrm{rk}_{\mathcal{F}}(A) := \lim_{\mathcal{W}} \frac{\mathrm{rk}_{\mathcal{W}}(A)}{\dim(\mathcal{W})} = \inf_{\mathcal{V}\in\mathcal{F}(S)\setminus\{0\}} \frac{\mathrm{rk}_{\mathcal{V}}(A)}{\dim(\mathcal{V})}.$$

Here, $\dim(\mathcal{V})$ and $\dim(\mathcal{W})$ denote the corresponding ranks as free modules, and $\lim_{\mathcal{W}}$ means convergence when $\mathcal{W}$ becomes more and more right invariant ([JiLi21, Definition 6.2]). If we take $\mathcal{W}_n = \sum_{i=0}^{n-1} Rt^i$, which is free of rank n, one can show that for every $\mathcal{V}$ in $\mathcal{F}(S)$ and every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\mathcal{W}_n$ is $(\mathcal{V}, \epsilon)$-invariant, what implies from the definition of the previous limit that for every matrix A over $R[t;\tau]$

$$\exists \lim_{n\to\infty} \frac{\mathrm{rk}_{\mathcal{W}_n}(A)}{\dim(\mathcal{W}_n)} = \lim_{\mathcal{W}} \frac{\mathrm{rk}_{\mathcal{W}}(A)}{\dim(\mathcal{W})} = \mathrm{rk}_{\mathcal{F}}(A)$$

As in their proof of [JiLi21, Lemma 9.3 & Proposition 9.4] (in the case $\{f_i\}_i = \{t^i\}_i$) about the relation between their construction and the natural transcendental extension introduced by A. Jaikin-Zapirain in [Jai19], one can show that then

$$\exists \lim_{n\to\infty} \frac{\mathrm{rk}(\phi_n^A)}{n} = \lim_{n\to\infty} \frac{\mathrm{rk}_{\mathcal{W}_n}(A)}{\dim(\mathcal{W}_n)},$$

from where the limit exists and equals $\mathrm{rk}_{\mathcal{F}}(A)$. Thus $\widetilde{\mathrm{rk}} = \mathrm{rk}_{\mathcal{F}}$ as a rank function on $R[t;\tau]$, and therefore on S by the previous remark.

Summing up, the following Sylvester rank function is always defined.

Definition 1.4.20. Let τ be an automorphism of a ring R and rk a τ-compatible Sylvester matrix rank function on R. The *natural transcendental extension of* rk to $R[t;\tau]$ is the Sylvester matrix rank function $\widetilde{\mathrm{rk}}$ on $R[t;\tau]$ defined, for every matrix A over $R[t;\tau]$, as the limit

$$\widetilde{\mathrm{rk}}(A) := \lim_{k\to\infty} \widetilde{\mathrm{rk}}_k(A).$$

Its extension to $R[t^{\pm 1};\tau]$ is called *natural transcendental extension of* rk to $R[t^{\pm 1};\tau]$.

We shall not give here a proof of its existence in the general setting, but with the purpose of making this document more or less self-contained we show in the next section how $\widetilde{\mathrm{rk}}$ (and its associated module rank function) is constructed for exact Sylvester rank functions, giving several characterizations and properties of $\widetilde{\mathrm{rk}}$ in this particular case. Later in the document, we shall also mention the case in which they come (in the sense of Definition 1.4.5) from epic division rings or $*$-regular rings after introducing these topics in Section 3.1 and Section 4.1, respectively.

The following is also an important enough remark to be kept as a lemma. Let rk be a τ-compatible Sylvester matrix rank function on R with natural transcendental extension $\widetilde{\mathrm{rk}}$. Then τ extends to an automorphism of $\mathrm{Mat}_n(R)$, and $\frac{1}{n}$ rk defines a τ-compatible Sylvester matrix rank function on $\mathrm{Mat}_n(R)$. Thus, the natural transcendental extension of $\frac{1}{n}$ rk is defined on $\mathrm{Mat}_n(R)[t;\tau]$ (or even on $\mathrm{Mat}_n(R)[t^{\pm 1};\tau]$). We show in the next lemma that the natural transcendental extension of $\frac{1}{n}$ rk is related to $\widetilde{\mathrm{rk}}$ via the ring isomorphism

$$\varphi_n : \mathrm{Mat}_n(R)[t;\tau] \to \mathrm{Mat}_n(R[t;\tau])$$

that sends the polynomial $p = \sum_l A_l t^l$ with $A_l = (a_{ij}^{(l)})$ to the matrix $A = (p_{ij})$ with $p_{ij} = \sum_l a_{ij}^{(l)} t^l$. The universal property of Ore localization (cf. [GW04, Proposition 6.3]) allows us to extend this to a ring homomorphism φ_n making the following diagram commutative

$$\begin{tikzcd} \mathrm{Mat}_n(R)[t;\tau] \arrow[r, "\varphi_n"] \arrow[d, hook] & \mathrm{Mat}_n(R[t;\tau]) \arrow[d, hook] \\ \mathrm{Mat}_n(R)[t^{\pm 1};\tau] \arrow[r, "\varphi_n"] & \mathrm{Mat}_n(R[t^{\pm 1};\tau]). \end{tikzcd} \tag{1.3}$$

This extension is again an isomorphism and sends pt^{-k}, for $p \in \mathrm{Mat}_n(R)[t;\tau]$ and a positive integer k, to $\varphi_n(p)\varphi_n(t)^{-k} = \varphi_n(p)(I_n t^{-k})$.

Lemma 1.4.21. *Let τ denote both an automorphism of a ring R and its extension to an automorphim of $S = \mathrm{Mat}_n(R)$. Assume that rk is a τ-compatible Sylvester matrix rank function with natural transcendental extension $\widetilde{\mathrm{rk}}$. If we set $\mathrm{rk}' = \frac{1}{n}\,\mathrm{rk}$, then*

$$\widetilde{\mathrm{rk}'} = \varphi_n^{\sharp}\left(\frac{1}{n}\,\widetilde{\mathrm{rk}}\right)$$

This expression is valid both in $S[t;\tau]$ and in $S[t^{\pm 1};\tau]$.

Proof. Let A be a $k \times l$ matrix over $S[t;\tau]$. On the one hand, since rk' is τ-compatible, we can define its i^{th} extension $\widetilde{\mathrm{rk}}'_i$ to $S[t;\tau]$, which is given by

$$\widetilde{\mathrm{rk}}'_i(A) = \frac{\mathrm{rk}'(\phi^A_{S,i})}{i}$$

where

$$\phi^A_{S,i} : \left({}^{S[t;\tau]}\big/_{S[t;\tau]t^i}\right)^k \longrightarrow \left({}^{S[t;\tau]}\big/_{S[t;\tau]t^i}\right)^l$$

is given by right multiplication by A. Since $\mathrm{rk}' = \frac{1}{n}\,\mathrm{rk}$, computing $\mathrm{rk}'(\phi^A_{S,i})$ amounts to compute the matrix associated to $\phi^A_{S,i}$, which is a $ki \times li$ matrix over S, watch it as an $nki \times nli$ matrix A_1 over R, and calculate $\frac{1}{n}\,\mathrm{rk}(A_1)$.

On the other hand, $\varphi_n(A)$ is a $k \times l$ matrix over $\mathrm{Mat}_n(R[t;\tau])$. If $\widetilde{\mathrm{rk}}_i$ denotes the i^{th} extension of rk to $R[t;\tau]$, in this latter ring we have a rank $\frac{1}{n}\,\widetilde{\mathrm{rk}}_i$. To compute $\frac{1}{n}\,\widetilde{\mathrm{rk}}_i(\varphi_n(A))$, we watch $\varphi_n(A)$ as an $nk \times nl$ matrix over $R[t;\tau]$, consider

$$\phi^{\varphi_n(A)}_{R,i} : \left({}^{R[t;\tau]}\big/_{R[t;\tau]t^i}\right)^{nk} \longrightarrow \left({}^{R[t;\tau]}\big/_{R[t;\tau]t^i}\right)^{nl}$$

given by right multiplication by $\varphi_n(A)$, take the matrix associated to $\phi^{\varphi_n(A)}_{R,i}$, which is an $nki \times nli$ matrix A_2 over R, and compute $\frac{1}{ni}\,\mathrm{rk}(A_2)$.

One can carefully check that, over R, these two $nki \times nli$ matrices A_1 and A_2 are equivalent to each other, and therefore

$$\begin{aligned} \widetilde{\mathrm{rk}}'_i(A) &= \frac{\mathrm{rk}'(\phi^A_{S,i})}{i} = \frac{1}{ni}\,\mathrm{rk}(A_1) = \frac{1}{ni}\,\mathrm{rk}(A_2) \\ &= \frac{1}{n}\,\widetilde{\mathrm{rk}}_i(\varphi_n(A)) = \varphi_n^{\sharp}\left(\frac{1}{n}\,\widetilde{\mathrm{rk}}_i\right)(A). \end{aligned}$$

Since the equality holds for every A, $\widetilde{\mathrm{rk}'_i} = \varphi_n^\sharp\left(\frac{1}{n}\widetilde{\mathrm{rk}}_i\right)$. Thus,

$$\begin{aligned}\varphi_n^\sharp\left(\frac{1}{n}\widetilde{\mathrm{rk}}\right)(A) &= \left(\frac{1}{n}\widetilde{\mathrm{rk}}\right)(\varphi_n(A)) \overset{*}{=} \frac{\widetilde{\mathrm{rk}}(\varphi_n(A))}{n} \\ &= \frac{\lim_{i\to\infty}\widetilde{\mathrm{rk}}_i(\varphi_n(A))}{n} \overset{**}{=} \lim_{i\to\infty}\frac{\widetilde{\mathrm{rk}}_i(\varphi_n(A))}{n} \\ &\overset{*}{=} \lim_{i\to\infty}\left(\frac{1}{n}\widetilde{\mathrm{rk}}_i\right)(\varphi_n(A)) = \lim_{i\to\infty}\varphi_n^\sharp\left(\frac{1}{n}\widetilde{\mathrm{rk}}_i\right)(A) \\ &= \lim_{i\to\infty}\widetilde{\mathrm{rk}'_i}(A)\end{aligned}$$

Here, after the first $(*)$ we see the $k \times l$ matrix $\varphi_n(A)$ on $\mathrm{Mat}_n(R[t;\tau])$ as an $nk \times nl$ matrix on $R[t;\tau]$, and after the second $(*)$ we consider it again on $\mathrm{Mat}_n(R[t;\tau])$. The limit preceding $(**)$ exists because $\widetilde{\mathrm{rk}}$ exists, and the latter equality follows from our previous discussion. This means precisely that $\varphi_n^\sharp\left(\frac{1}{n}\widetilde{\mathrm{rk}}\right)$ is the natural transcendental extension of rk' to $S[t;\tau]$.

Furthermore, we can consider $\widetilde{\mathrm{rk}}$ on $R[t^{\pm1};\tau]$ and hence $\varphi_n^\sharp\left(\frac{1}{n}\widetilde{\mathrm{rk}}\right)$ as a rank on $S[t^{\pm1};\tau]$. By commutativity of diagram (1.3), we have just proved that its restriction to $S[t;\tau]$ coincides with $\widetilde{\mathrm{rk}'}$. Thus, by Remark 1.4.19, it is the natural extension of rk' to $S[t^{\pm1};\tau]$. □

Given a τ-compatible rank function rk, Lemma 1.4.21 sometimes allows us to reduce claims about $\widetilde{\mathrm{rk}}(A)$ for any matrix A over $R[t;\tau]$ to claims about $\widetilde{\mathrm{rk}}(a)$ for just elements $a \in R[t;\tau]$. Indeed, since a rank does not change when adding or deleting rows and columns of zeros, as Properties 1.2.2(4.) shows, we can always restrict our attention to square matrices. Assume now that we prove a result for any natural extension and for any element in the corresponding ring, and let rk be a concrete Sylvester matrix rank function on R with natural extension $\widetilde{\mathrm{rk}}$. Lemma 1.4.21 tells us then that the property is also satisfied by $\varphi_n^\sharp\left(\frac{1}{n}\widetilde{\mathrm{rk}}\right)$ on elements of $\mathrm{Mat}_n(R)$, and this can be used to derive similar properties for $\widetilde{\mathrm{rk}}$ on $n \times n$ matrices.

As we already mentioned, the upcoming section is devoted to the study of the natural transcendental extension of a τ-compatible rank whose associated dimension function is exact. In addition, we develop there further properties and characterizations for $\widetilde{\mathrm{rk}}$ when the ring R under consideration is regular.

1.5 The natural transcendental extension of exact ranks

In this section, we show how the exactness condition on dim ensures the existence of the natural transcendental extension of its associated rank rk. Throughout this section, we fix a ring R, an automorphism τ of R and a τ-compatible exact Sylvester module rank function dim on R with associated Sylvester matrix rank function rk. In particular, dim actually defines a normalized length function by means of Proposition 1.4.4.

Observe first that the τ-compatibility is preserved when we extend dim to all R-modules. Indeed, observe that every surjection $\varphi_1 : L \twoheadrightarrow N$ gives rise to a surjection $\psi_1 : tL \twoheadrightarrow tN$ defined by $\psi_1(tl) = t\varphi_1(l)$ and, conversely, every surjection $\varphi_2 : L' \twoheadrightarrow tM$ gives rise to a surjection $\psi_2 : t^{-1}L' \twoheadrightarrow M$ sending $t^{-1}l \to m$ if $\varphi_2(l) = tm$. Since L is finitely presented if and only if t^kL is finitely presented for every k, we can deduce from the τ-compatibility on finitely presented modules and according to Proposition 1.4.4, that for a finitely generated left R-module N, we have $\dim(tN) = \dim(N)$.

Now, if M is a left R-module and $N \leq M$ is a finitely generated R-submodule, then $tN \leq tM$ is finitely generated, and conversely, if $N' \leq tM$ is finitely generated, then $N' = tN$ for some finitely generated $N \leq M$. Thus, the τ-compatibility on finitely generated modules and the definition on Proposition 1.4.4 imply that $\dim(tM) = \dim(M)$.

Under these assumptions on dim, the natural transcendental extension is related to the notion of algebraic entropy introduced in [Vir19A], more generally defined in the context of crossed products ($R[t^{\pm 1};\tau]$ is an instance of crossed product $R * \mathbb{Z}$, as we will recall in Section 3.4).

In this sense, our notion of τ-compatibility for the extension of dim is the same defined on [Vir19A, Definition 3.6]. Moreover, since the length function dim is normalized, the dimension of an n-generated module is at most n, and hence the domain of definition of the entropy as in [Vir19A, Theorem B] is the family of all left $R[t^{\pm 1};\tau]$-modules. Thus, from [Vir19A, Definition 4.3] applied to $R[t^{\pm 1};\tau] = R * \mathbb{Z}$ and the Følner sequence $F_n = \{1, t, \dots, t^{n-1}\}$ of $\mathbb{Z}$, and from [Vir19A, Theorem B], we deduce the next result.

Proposition 1.5.1. *Define, for every $R[t^{\pm 1};\tau]$-module M,*

$$\widetilde{\dim}(M) = \sup\{E_{M,N} : N \textit{ is an } R\textit{-submodule of } M \textit{ and } \dim(N) < \infty\},$$

where

$$E_{M,N} = \lim_{i\to\infty} \frac{\dim(N + tN + \cdots + t^{i-1}N)}{i}.$$

Then $\widetilde{\dim}$ is a well-defined normalized length function on $R[t^{\pm 1};\tau]$.

We are going to show that the Sylvester matrix rank function associated to $\widetilde{\dim}$ is actually the natural transcendental extension of rk, which is not only defined on $R[t;\tau]$ but on $R[t^{\pm 1};\tau]$. For this, we need the following results, corresponding to [Jai19, Lemma 7.2, Lemma 7.3 & Proposition 7.4] but adapted to the skew case. For this purpose, let Q_n be the set of polynomials in $R[t;\tau]$ of degree at most n.

Lemma 1.5.2. *Let M be a finitely generated $R[t^{\pm 1};\tau]$-module with generator set $\{m_1, \dots, m_k\}$, and let V be the R-submodule of M generated by this set. Then*

$$\widetilde{\dim} M = E_{M,V} = \lim_{i\to\infty} \frac{\dim(V + tV + \cdots + t^{i-1}V)}{i}$$

Proof. Let N be any R-submodule of M with $\dim(N) < \infty$. Since τ is an automorphism and any element x of M has the form $x = \sum_{i=1}^{k} p_i m_i$ for some $p_1, \ldots, p_k \in R[t^{\pm 1}; \tau]$, we can always find n such that $x \in t^{-n} Q_{2n} V$, which is a finitely generated (hence with finite dimension) R-submodule of M. Now, dim is a normalized length function, so $\dim(N)$ is the supremum of the dimensions of its finitely generated R-submodules, each of which is contained in $t^{-n} Q_{2n} V \cap N$ for an appropriate choice of n. Therefore,

$$\dim(N) = \sup_n \{\dim(t^{-n} Q_{2n} V \cap N)\}$$

and reasoning similarly,

$$\dim(N + \cdots + t^{i-1} N) = \sup_n \{\dim(t^{-n} Q_{2n} V \cap N + \cdots + t^{i-1}(t^{-n} Q_{2n} V \cap N))\}.$$

Therefore, since the limit $E_{M,N}$ exists and by the τ-compatibility of dim and the surjectivity of the canonical R-homomorphism $\bigoplus t^i N \twoheadrightarrow \sum t^i N$ it is bounded by $\dim(N) < \infty$, we can interchange limit and supremum to conclude that

$$E_{M,N} = \sup_n E_{M, t^{-n} Q_{2n} V \cap N} \leq \sup_n E_{M, t^{-n} Q_{2n} V}.$$

Hence, using τ-compatibility again,

$$\widetilde{\dim}(M) = \sup_n E_{M, t^{-n} Q_{2n} V} = \sup_n E_{M, Q_{2n} V}.$$

Finally, since τ is an automorphism, we have $Rt = tR$ and we can write $Q_{2n} = \sum_{j=0}^{2n} t^j R$, from where $Q_{2n} V = \sum_{j=0}^{2n} t^j V$ and

$$Q_{2n} V + \cdots + t^{i-1} Q_{2n} V = \sum_{j=0}^{2n+i-1} t^j V = \sum_{j=0}^{i-1} t^j V + \sum_{j=i}^{2n+i-1} t^j V.$$

Therefore, for every fixed n and taking limits on i in the previous expression we can see that $E_{M, Q_{2n} V} = E_{M,V}$, from where $\widetilde{\dim}(M) = E_{M,V}$, as claimed. □

Lemma 1.5.3. *Let* rk' *be the Sylvester matrix rank function on* $R[t^{\pm 1}; \tau]$ *associated to* $\widetilde{\dim}$. *Then, for any* $n \times m$ *matrix* A *over* $R[t; \tau]$, *we have*

$$\mathrm{rk}'(A) = \lim_{i \to \infty} \frac{\dim((Q_{i-1})^n A)}{i}.$$

Proof. Since $\widetilde{\dim}$ is additive on exact sequences, observe that

$$\mathrm{rk}'(A) = m - \widetilde{\dim}\left(R[t^{\pm 1}; \tau]^m \big/ R[t^{\pm 1}; \tau]^n A \right) = \widetilde{\dim}(R[t^{\pm 1}; \tau]^n A).$$

$R[t^{\pm 1}; \tau]^n A$ is the $R[t^{\pm 1}; \tau]$-module generated by the n rows of A, and the R-submodule generated by them is $R^n A$. From the equality $\sum_{j=0}^{i-1} t^j (R^n A) = (Q_{i-1})^n A$ and Lemma 1.5.2, we deduce that

$$\mathrm{rk}'(A) = \widetilde{\dim}(R[t^{\pm 1}; \tau]^n A) = E_{M, R^n A} = \lim_{i \to \infty} \frac{\dim((Q_{i-1})^n A)}{i}.$$

□

With these two lemmas, we can now prove that rk' is actually the natural transcendental extension of rk.

Proposition 1.5.4. *Let* rk' *be the Sylvester matrix rank function on* $R[t^{\pm 1};\tau]$ *associated to* $\widetilde{\dim}$. *Then, for every matrix* A *over* $R[t;\tau]$,

$$\mathrm{rk}'(A) = \lim_{i\to\infty} \widetilde{\mathrm{rk}}_i(A)$$

where $\widetilde{\mathrm{rk}}_i$ *is the* i^{th} *extension of* rk *to* $R[t;\tau]$, *i.e.,* rk' *is the natural transcendental extension of* rk *to* $R[t^{\pm 1};\tau]$.

Proof. Let us first illustrate the result for an element $p = a_0 + \cdots + a_s t^s$. Recall that $\widetilde{\mathrm{rk}}_i(p) = \frac{1}{i}\,\mathrm{rk}(\phi_i^p)$, and let B be the matrix associated to ϕ_i^p with respect to the canonical basis $\{1 + R[t;\tau]t^i, \ldots, t^{i-1} + R[t;\tau]t^i\}$. By definition, for any $q = r_0 + \cdots + r_k t^k$, the first i coefficients of qp are given precisely by $(r_0, \ldots, r_{i-1})B$ (setting $r_j = 0$ if $j > k$). Define the R-homomorphisms

$$\eta_k : Q_k p \to R^i B$$

sending qp to its first i coefficients. For every $i \geq s+1$, η_{i-1} is surjective, since each $(r_0, \ldots, r_{i-1})B$ has a preimage $(\sum_{j=0}^{i-1} r_j t^j)p$, and η_{i-s-1} is injective, since $Q_{i-s-1}p \subseteq Q_{i-1}$. Additivity of dim tells us then that

$$\dim(Q_{i-s-1}p) \leq \dim(R^i B) \leq \dim(Q_{i-1}p).$$

Since from the additivity on short exact sequences we also see that

$$\frac{\dim(R^i B)}{i} = \frac{i - \dim(R^i/R^i B)}{i} = \frac{\mathrm{rk}(B)}{i} = \widetilde{\mathrm{rk}}_i(p)$$

we have proved that for $i \geq s+1$,

$$\frac{\dim(Q_{i-s-1}p)}{i} \leq \widetilde{\mathrm{rk}}_i(p) \leq \frac{\dim(Q_{i-1}p)}{i}.$$

Taking limits on i and using Lemma 1.5.3, we conclude that

$$\mathrm{rk}'(p) = \lim_{i\to\infty} \widetilde{\mathrm{rk}}_i(p).$$

The result for matrices is obtained similarly. If A is an $n \times m$ matrix over $R[t;\tau]$ and C is the matrix associated to ϕ_i^A with respect to the canonical basis in each factor, then for any $(q_1, \ldots, q_n) \in R[t;\tau]^n$, with $q_l = \sum_u r_u^{(l)} t^u$, the first i coefficients of the j^{th} entry of $(q_1, \ldots, q_n)A$ are the i coefficients from position $(j-1)i+1$ to ji of

$$(r_0^{(1)}, \ldots, r_{i-1}^{(1)}, \ldots, r_0^{(n)}, \ldots, r_{i-1}^{(n)})C,$$

Therefore, we can analogously define R-maps $Q_k^n A \to R^{in}C$ and find an appropriate s such that for every $i \geq s+1$

$$\frac{\dim((Q_{i-s-1})^n A)}{i} \leq \widetilde{\mathrm{rk}}_i(A) \leq \frac{\dim((Q_{i-1})^n A)}{i}.$$

Hence, the result follows from Lemma 1.5.3 as above. $\square$

For the rest of the section, assume that $R = \mathcal{U}$ is regular, and recall that every Sylvester module rank function on $\mathcal{U}$ is exact, so the previous results hold for the natural transcendental extension of every τ-compatible rank rk. Under the regularity assumption, we can give another explicit formula for $\widetilde{\dim}(I)$ for every left ideal of $R[t^{\pm 1};\tau]$, and a characterization of the natural transcendental extension. These results correspond to [Jai19, Proposition 7.6& Proposition 7.7] for skew Laurent polynomials.

Proposition 1.5.5. *For every left ideal I of $\mathcal{U}[t^{\pm 1};\tau]$,*

$$\widetilde{\dim}(I) = \sup\left\{ \mathrm{rk}(a_0) : a_0 \in \mathcal{U} \text{ and } \exists n \geq 0, \exists a_1, \ldots, a_n \in \mathcal{U} \text{ s.t. } \sum_{i=0}^{n} a_i t^i \in I \right\}$$

Proof. Define $P_i = Q_i \cap I$, i.e., the set of polynomials in $\mathcal{U}[t;\tau]$ of degree at most i contained in I, and set $M = \mathcal{U}[t^{\pm 1};\tau]/I$. By additivity of $\widetilde{\dim}$ in short exact sequences,

$$\widetilde{\dim}(I) = 1 - \widetilde{\dim}(M).$$

Now M is generated by $1 + I$, and the $\mathcal{U}$-submodule of M generated by this element is $V = (\mathcal{U} + I)/I$. Thus, one can check that $t^k V = (t^k \mathcal{U} + I)/I$ and

$$V + tV + \cdots + t^{i-1} V = (\mathcal{U} + t\mathcal{U} + \cdots + t^{i-1}\mathcal{U} + I)/I = (Q_{i-1} + I)/I.$$

Since by the second isomorphism theorem, $(Q_{i-1} + I)/I \cong Q_{i-1}/(Q_{i-1} \cap I) = Q_{i-1}/P_{i-1}$, we deduce from Lemma 1.5.2 and by additivity of dim that

$$\widetilde{\dim}(M) = E_{M,V} = \lim_{i \to \infty} \frac{\dim(Q_{i-1}/P_{i-1})}{i} = \lim_{i \to \infty} \frac{\dim(Q_{i-1}) - \dim(P_{i-1})}{i}.$$

As a left R-module, $Q_{i-1} = \mathcal{U} \oplus \cdots \oplus t^{i-1}\mathcal{U}$, and hence by τ-compatibility we deduce that $\dim(Q_{i-1}) = i$. Adding everything up,

$$\widetilde{\dim}(I) = 1 - \lim_{i \to \infty} \frac{i - \dim(P_{i-1})}{i} = \lim_{i \to \infty} \frac{\dim(P_{i-1})}{i}.$$

Now, observe that for every left $\mathcal{U}$-module N and for every integer k, we can define a left $\mathcal{U}$-module Nt^k whose elements are of the form xt^k for $x \in N$ and with natural sum and $\mathcal{U}$-product. N is naturally isomorphic to Nt^k, so $\dim(N) = \dim(Nt^k)$. In particular, if we consider the left R-module

$$\tau^{-1}(P_i) := \left\{ \sum_{j=0}^{i} \tau^{-1}(a_j) t^j : \sum_{j=0}^{i} a_j t^j \in P_i \right\}$$

we see that $\tau^{-1}(P_i) \cong t^{-1}(P_i t)$ as $\mathcal{U}$-modules and hence by τ-compatibility we obtain that $\dim(\tau^{-1}(P_i)) = \dim(P_i)$. Moreover, if $p = a_0 + \cdots + a_{i-1}t^{i-1} \in P_{i-1}$, then $tp = \tau(a_0)t + \cdots + \tau(a_{i-1})t^i \in P_i$ because I is a left ideal, and hence by definition $a_0 t + \cdots +$

$a_{i-1}t^i = pt \in \tau^{-1}(P_i)$. In other words, right multiplication by t sends P_{i-1} into $\tau^{-1}(P_i)$ and induces injective $\mathcal{U}$-homomorphisms

$$P_{i-1}\big/P_{i-2} \longrightarrow \tau^{-1}(P_i)\big/\tau^{-1}(P_{i-1})$$

for every $i \geq 2$. If we set $P_{-1} = 0$, the injectivity also holds for $i = 1$. From the additivity of dim, the injectivity of the previous $\mathcal{U}$-homomorphisms and τ-compatibility, we deduce for every $i \geq 1$ that

$$\begin{aligned}\dim(P_{i-1}) - \dim(P_{i-2}) &= \dim(P_{i-1}/P_{i-2}) \leq \dim(\tau^{-1}(P_i)/\tau^{-1}(P_{i-1})) \\ &= \dim(\tau^{-1}(P_i)) - \dim(\tau^{-1}(P_{i-1})) \\ &= \dim(P_i) - \dim(P_{i-1}),\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\widetilde{\dim}(I) &= \lim_{i\to\infty} \frac{\dim(P_i)}{i+1} = \lim_{i\to\infty} \frac{\sum_{j=0}^{i} \dim(P_j) - \dim(P_{j-1})}{i+1} \\ &\overset{*}{=} \lim_{i\to\infty} \dim(P_i) - \dim(P_{i-1}) \overset{**}{=} \lim_{i\to\infty} \dim(P_i) - \dim(tP_{i-1}) \\ &= \lim_{i\to\infty} \dim(P_i/tP_{i-1}),\end{aligned}$$

where $(**)$ holds by τ-compatibility and $(*)$ holds because the sequence $\{a_j\}_{j\geq 1}$ given by $a_j = \dim(P_{j-1}) - \dim(P_{j-2})$ is monotonically increasing, the limit $\lim_i \frac{1}{i+1}\sum_{j=1}^{i+1} a_j$ exists by hypothesis, and therefore it must coincide with $\lim_i a_{i+1}$.

Furthermore, since I is a left ideal, the sets

$$T_k := \left\{ a_0 \in \mathcal{U} : \exists a_1, \ldots, a_k \in \mathcal{U} \text{ s.t. } \sum_{i=0}^{k} a_i t^i \in I \right\}$$

are left ideals of $\mathcal{U}$, and the map $P_k/tP_{k-1} \to T_k$ sending $p + tP_{k-1}$ to the constant term of p defines a $\mathcal{U}$-isomorphism. Thus,

$$\widetilde{\dim}(I) = \lim_{k\to\infty} \dim(T_k).$$

It is now when we are going to use the regularity of $\mathcal{U}$: every finitely generated $\mathcal{U}$-submodule N of T_k is a finitely generated left ideal of $\mathcal{U}$, and hence principal by Proposition 1.3.3(ii), i.e., $N = \mathcal{U}a_0$ for some $a_0 \in T_k$. Furthermore, by additivity of dim, $\dim(N) = \mathrm{rk}(a_0)$, and since dim is a normalized length function,

$$\begin{aligned}\dim(T_k) &= \sup\{\dim(N) : N \text{ finitely generated and } N \leq T_k\} \\ &= \sup\{\mathrm{rk}(a_0) : a_0 \in T_k\} \\ &= \sup\left\{ \mathrm{rk}(a_0) : a_0 \in \mathcal{U} \text{ and } \exists a_1, \ldots, a_k \in \mathcal{U} \text{ s.t. } \sum_{i=0}^{k} a_i t^i \in I \right\},\end{aligned}$$

Finally, since $\dim(T_k)$ is the supremum of the ranks of elements $a_0 \in \mathcal{U}$ which are constant terms of polynomials in I of degree at most k, $\lim_k \dim(T_k)$ is the supremum of the ranks of elements $a_0 \in \mathcal{U}$ which are constant terms of some polynomial in I, i.e.,

$$\widetilde{\dim}(I) = \sup\left\{ \operatorname{rk}(a_0) : a_0 \in \mathcal{U} \text{ and } \exists n \geq 0, \exists a_1, \ldots, a_n \in \mathcal{U} \text{ s.t. } \sum_{i=0}^{n} a_i t^i \in I \right\}$$

as claimed. □

The next proposition gives a characterization of $\widetilde{\operatorname{rk}}$ for regular rings.

Proposition 1.5.6. *If* rk^* *is a Sylvester matrix rank function on* $\mathcal{U}[t^{\pm 1};\tau]$ *that extends* rk, *then* $\operatorname{rk}^* = \widetilde{\operatorname{rk}}$ *if and only if for every matrix* $A \in \operatorname{Mat}_n(\mathcal{U})$,

$$\operatorname{rk}^*(I_n + At) = n$$

Proof. By definition, since $A' = I_n + At \in \operatorname{Mat}(\mathcal{U}[t;\tau])$,

$$\widetilde{\operatorname{rk}}(A') = \lim_{k\to\infty} \frac{\operatorname{rk}(\phi_k^{A'})}{k}$$

Setting $M_k^n = (\mathcal{U}[t;\tau]/\mathcal{U}[t;\tau]t^k)^n$, one can check that $\phi_1^{A'} = \operatorname{id}_{M_1^n}$ and that for every $k \geq 2$, $\phi_k^{A'}$ is a $\mathcal{U}$-automorphism of M_k^n with inverse $\phi_k^{B_k}$, where

$$B_k = I_n - At + \sum_{i=2}^{k-1} (-1)^i A\tau(A)\cdots\tau^{i-1}(A)t^i$$

Indeed, $A'B_k = B_kA' = I_n + (-1)^{k-1}A\tau(A)\cdots\tau^{k-1}(A)t^k$, and hence we have $\phi_k^{A'}\phi_k^{B_k} = \phi_k^{B_k}\phi_k^{A'} = \operatorname{id}_{M_k^n}$. Therefore,

$$\widetilde{\operatorname{rk}}(A') = \lim_{k\to\infty} \frac{\operatorname{rk}(\phi_k^{A'})}{k} = \lim_{k\to\infty} \frac{nk}{k} = n.$$

This gives us the "only if" part. For the "if" part, let us first show that the given property extends to matrices of the form $M = I_n + A_1t + \cdots + A_kt^k$ for $A_i \in \operatorname{Mat}_n(\mathcal{U})$. Indeed, the result for $k = 1$ is our hypothesis, so let $k \geq 1$ and assume that we have already proved the result up to degree $k-1$. Write $M = I_n + A_1t + Bt^2$, where $B \in \operatorname{Mat}_n(\mathcal{U}[t;\tau])$ contains polynomials up to degree $k-2$. Then

$$\begin{aligned}
\operatorname{rk}^*(M) + n &= \operatorname{rk}^*\begin{pmatrix} M & 0 \\ 0 & I_n \end{pmatrix} \\
&= \operatorname{rk}^*\left(\begin{pmatrix} I_n & -I_nt^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} M & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_n & I_nt^{-1} \\ 0 & I_n \end{pmatrix}\right) \\
&= \operatorname{rk}^*\begin{pmatrix} M & Mt^{-1} - I_nt^{-1} \\ 0 & I_n \end{pmatrix} = \operatorname{rk}^*\begin{pmatrix} M & A_1 + Bt \\ 0 & I_n \end{pmatrix} \\
&= \operatorname{rk}^*\left(\begin{pmatrix} I_n & -A_1 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} M & A_1 + Bt \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_n & 0 \\ -I_nt & I_n \end{pmatrix}\right) \\
&= \operatorname{rk}^*\begin{pmatrix} I_n + A_1t & Bt \\ -I_nt & I_n \end{pmatrix} = \operatorname{rk}^*\left(I_{2n} + \begin{pmatrix} A_1 & B \\ -I_n & 0 \end{pmatrix} t\right)
\end{aligned}$$

The latter matrix has size $2n \times 2n$ and contains polynomials of degree $\leq k-1$. Hence the inductive hypothesis tells us that its rank is $2n$, and we deduce that

$$\mathrm{rk}^*(I_n + A_1 t + \cdots + A_k t^k) = n$$

We want to show that $\mathrm{rk}^*(A) = \widetilde{\mathrm{rk}}(A)$ for every matrix $A \in \mathcal{U}[t^{\pm 1};\tau]$. Let us show the result for elements $p \in \mathcal{U}[t^{\pm 1};\tau]$, and for that purpose consider the left annihilator of p in $\mathcal{U}[t^{\pm 1};\tau]$

$$\mathrm{Ann}_l(p) = \{q \in \mathcal{U}[t^{\pm 1};\tau] : qp = 0\},$$

which is a left ideal of $\mathcal{U}[t^{\pm 1};\tau]$ satisfying $\mathcal{U}[t^{\pm 1};\tau]/\,\mathrm{Ann}_l(p) \cong \mathcal{U}[t^{\pm 1};\tau]p$. From the additivity of $\widetilde{\dim}$ we deduce that

$$\begin{aligned}
\widetilde{\mathrm{rk}}(p) &= 1 - \widetilde{\dim}(\mathcal{U}[t^{\pm 1};\tau]/\mathcal{U}[t^{\pm 1};\tau]p) = \widetilde{\dim}(\mathcal{U}[t^{\pm 1};\tau]p) \\
&= \widetilde{\dim}(\mathcal{U}[t^{\pm 1};\tau]/\,\mathrm{Ann}_l(p)) = 1 - \widetilde{\dim}(\mathrm{Ann}_l(p))
\end{aligned}$$

Moreover, Proposition 1.5.5 tells us that, for every $\epsilon > 0$, we can find polynomials $q_1 = b_0 + b_1 t + \cdots + b_k t^k \in \mathcal{U}[t^{\pm 1};\tau]p$, $q_2 = c_0 + c_1 t + \cdots + c_l t^l \in \mathrm{Ann}_l(p)$ such that

$$\widetilde{\dim}(\mathcal{U}[t^{\pm 1};\tau]p) \leq \mathrm{rk}(b_0) + \epsilon \quad \text{and} \quad \widetilde{\dim}(\mathrm{Ann}_l(p)) \leq \mathrm{rk}(c_0) + \epsilon,$$

and thus from the previous equalities $1 - \mathrm{rk}(c_0) - \epsilon \leq \widetilde{\mathrm{rk}}(p) \leq \mathrm{rk}(b_0) + \epsilon$. Now, since $\mathcal{U}$ is regular, we can find $x, y \in \mathcal{U}$ such that $b_0 x b_0 = b_0$ and $c_0 y c_0 = c_0$. Hence, on the one hand, since $q_1 = q'p$ for some $q' \in \mathcal{U}[t^{\pm 1};\tau]$,

$$\begin{aligned}
\mathrm{rk}^*(p) &\geq \mathrm{rk}^*(q_1) \geq \mathrm{rk}^*(b_0 x q_1) = \mathrm{rk}^*(b_0 + b_0 x b_1 t + \cdots + b_0 x b_k t^k) \\
&= \mathrm{rk}^*(b_0(1 + x b_1 t + \cdots + x b_k t^k)) \overset{*}{=} \mathrm{rk}^*(b_0) \overset{**}{=} \mathrm{rk}(b_0) \geq \widetilde{\mathrm{rk}}(p) - \epsilon,
\end{aligned}$$

where $(*)$ follows because $\mathrm{rk}^*(1 + x b_1 t + \cdots + x b_k t^k) = 1$ as we saw before and using Properties 1.2.2(6.), and $(**)$ holds because rk^* extends rk. Similarly, on the other hand, since $q_2 p = 0$, we see from Properties 1.2.2(5.) that $\mathrm{rk}^*(p) \leq 1 - \mathrm{rk}^*(q_2)$ and consequently

$$\begin{aligned}
\mathrm{rk}^*(p) &\leq 1 - \mathrm{rk}^*(q_2) \leq 1 - \mathrm{rk}^*(c_0 y q_2) = 1 - \mathrm{rk}^*(c_0 + c_0 y c_1 t + \cdots + c_0 y c_l t^l) \\
&= 1 - \mathrm{rk}^*(c_0(1 + y c_1 t + \cdots + y c_l t^l)) \overset{*}{=} 1 - \mathrm{rk}^*(c_0) \overset{**}{=} 1 - \mathrm{rk}(c_0) \\
&\leq \widetilde{\mathrm{rk}}(p) + \epsilon,
\end{aligned}$$

where $(*)$ and $(**)$ are deduced as in the previous case. Since this is valid for every ϵ, we conclude that $\mathrm{rk}^*(p) = \widetilde{\mathrm{rk}}(p)$ for every element $p \in \mathcal{U}[t^{\pm 1};\tau]$.

Notice that to prove the result in general it suffices to show that equality holds for $m \times m$ matrices, since we can add rows and columns of zeros without changing the rank (Properties 1.2.2(4.)). Moreover, we can restrict our attention to matrices over $\mathcal{U}[t;\tau]$, since we can always multiply by the invertible matrix $I_m t^{-k}$ for a suitable k, operation that again leaves the rank unchanged (Properties 1.2.2(2.) & (6.)). Let φ_m be the isomorphism defined in diagram (1.3), and observe that $S = \mathrm{Mat}_m(\mathcal{U})$ is regular (Example 1.3.2(c)) and $\mathrm{rk}_m = \varphi_m^\sharp\left(\frac{1}{m}\,\mathrm{rk}^*\right)$ is a rank function on $S[t^{\pm 1};\tau]$ satisfying, for every matrix A_0 over S, the following:

1. Since $\varphi_m(A_0) = A_0$ considered as a matrix over $S[t^{\pm 1};\tau]$, we have

$$\begin{aligned} \mathrm{rk}_m(A_0) &= \varphi_m^\sharp\left(\frac{1}{m}\mathrm{rk}^*\right)(A_0) = \left(\frac{1}{m}\mathrm{rk}^*\right)(A_0) = \frac{\mathrm{rk}^*(A_0)}{m} \\ &\overset{*}{=} \frac{\mathrm{rk}(A_0)}{n} = \left(\frac{1}{m}\mathrm{rk}\right)(A_0) \end{aligned}$$

where $(*)$ holds because rk^* extends rk. Hence, rk_m extends $\frac{1}{m}\mathrm{rk}$.

2. If A_0 is square of size $n \times n$ and $I_{S,n}$ denotes the $n \times n$ identity matrix over S,

$$\begin{aligned} \mathrm{rk}_m(I_{S,n} + A_0 t) &= \varphi_m^\sharp\left(\frac{1}{m}\mathrm{rk}^*\right)(I_{S,n} + A_0 t) = \left(\frac{1}{m}\mathrm{rk}^*\right)(I_{S,n} + A_0 t) \\ &= \frac{\mathrm{rk}^*(I_{S,n} + A_0 t)}{m} \overset{*}{=} \frac{mn}{m} = n. \end{aligned}$$

Here, $(*)$ follows because in the preceding fraction we see $I_{S,n} + A_0 t$ as a matrix over $\mathcal{U}[t^{\pm 1};\tau]$, and hence $I_{S,n}$ becomes the identity matrix I_{mn} over $\mathcal{U}$. Thus, by hypothesis, $\mathrm{rk}^*(I_{S,n} + A_0 t) = nm$.

In other words, S, rk_m and $\frac{1}{m}\mathrm{rk}$ satisfy the hypothesis of the theorem. Thus, as we have already proved, rk_m must coincide with the natural transcendental extension of $\frac{1}{m}\mathrm{rk}$ on elements of $S[t^{\pm 1};\tau]$. But Lemma 1.4.21 tells us that the natural extension of $\frac{1}{m}\mathrm{rk}$ is $\varphi_m^\sharp\left(\frac{1}{m}\widetilde{\mathrm{rk}}\right)$ and consequently, for every element $A \in S[t;\tau]$, we have

$$\begin{aligned} \left(\tfrac{1}{m}\mathrm{rk}^*\right)(\varphi_m(A)) &= \varphi_m^\sharp\left(\tfrac{1}{m}\mathrm{rk}^*\right)(A) = \mathrm{rk}_m(A) \\ &= \varphi_m^\sharp\left(\tfrac{1}{m}\widetilde{\mathrm{rk}}\right)(A) = \left(\tfrac{1}{m}\widetilde{\mathrm{rk}}\right)(\varphi_m(A)). \end{aligned}$$

The bijectivity of φ_m gives then the equality $\left(\frac{1}{m}\mathrm{rk}^*\right)(B) = \left(\frac{1}{m}\widetilde{\mathrm{rk}}\right)(B)$ for every element $B \in \mathrm{Mat}_m(\mathcal{U}[t;\tau])$ and hence we have $\mathrm{rk}^*(B) = \widetilde{\mathrm{rk}}(B)$ for every square matrix on $\mathcal{U}[t;\tau]$. This finishes the proof. □

Once we introduce and describe in the next chapter the basic properties of the space of Sylvester matrix rank functions on a ring R, we will point out that, when $\mathcal{U}$ is regular, the natural transcendental extension of the rank rk is a regular rank function on $\mathcal{U}[t^{\pm 1};\tau]$.

Chapter 2

The space of Sylvester matrix rank functions

In the previous chapter we introduced different notions of rank functions and show how they are related to each other. As discussed during the introduction, the notion of Sylvester matrix (or module) rank function serves both as a classifying tool and as a common language to transcribe and tackle many different problems. This chapter explores the space $\mathbb{P}(R)$ of all Sylvester matrix rank functions that can be defined on a given ring R, introduce some of its basic properties and completely describes it for certain families of rings.

The latter part of classification and description comes from [JL20B], and relies heavily in the well-known structure of finitely generated modules — or equivalently finitely presented modules, since all the concrete examples we work with are left noetherian (cf. [Rot09, Corollary 3.19]) — for the families considered. This allows us, once we identify potential candidates for extreme points of the space, to actually check that they are extreme and, moreover, that every other rank function in the space is a unique (possibly countably infinite) convex combination of them.

The organization of the chapter is as follows. We introduce in Section 2.1 the space of Sylvester matrix rank functions and the first examples for which we can either describe it completely or relate the spaces associated to different rings. The rest of the sections are each dedicated to describe the space of rank functions on a particular family of rings. We start by studying the space of rank functions on a certain subfamily of left artinian primary rings in Section 2.2. This subfamily appears when dealing with quotients of Dedekind domains and skew Laurent polynomial rings $\mathcal{D}[t^{\pm 1};\tau]$, where $\mathcal{D}$ is a division ring and τ an automorphism of $\mathcal{D}$, and therefore allows us to get a partial picture of the space of rank functions for those two families, which are later studied in Section 2.3 and Section 2.5, respectively. In Section 2.4 we discuss the case of simple noetherian rings, which also appears naturally when dealing with Laurent polynomial rings.

2.1 The space $\mathbb{P}(R)$ and first examples

Let R be a ring. In this section we introduce and study the basic properties of $\mathbb{P}(R)$, the space of Sylvester rank functions on R, and present some basic examples of rings for which we can totally or partially describe $\mathbb{P}(R)$. Here, by "describing $\mathbb{P}(R)$" we mean either identifying the extreme points in $\mathbb{P}(R)$, so that any other rank function is a convex combination of those, or finding rings S and maps (of sets) $\mathbb{P}(S) \to \mathbb{P}(R)$ that we can show to be injective, surjective or bijective. However, although we shall not be especially concerned about further topological properties of the space, we need some definitions to introduce it properly.

Definition 2.1.1.

- A *directed set* $(I, \leq)$ consists of a set I together with a reflexive and transitive binary relation $\leq$ satisfying that, for each $i, j \in I$, there exists $k \in I$ such that $k \geq i$ and $k \geq j$.

- If X is a topological space, then a *net* in X is a map $I \to X$ where $(I, \leq)$ is a directed set. If the map sends $i \mapsto x_i$, we denote this net as usual by $\{x_i\}_{i \in I}$.

- We say that a net $\{x_i\}_i \in I$ in a topological space X *converges to* $x \in X$ if, for every neighborhood $U \subset X$ of x, there exists $i_0 \in I$ such that for every $i \geq i_0$, $x_i \in U$. We say that x is a *limit* of $\{x_i\}_{i \in I}$ and write $x_i \to x$.

Some authors also require the directed set to be antisymmetric, but we do not need this property for our purposes in this chapter.

The point about nets is that they allow us to reformulate some of the most important topological properties that we shall discuss here. The following properties can be found, for the particular case of Hausdorff spaces, in [Con14, Proposition 2.7.7 & Proposition 2.7.8].

Proposition 2.1.2. *Let X and Y be topological spaces. Then*

1. *The space X is Hausdorff if and only if every convergent net in X has a unique limit.*

2. *A subset $\mathcal{C}$ of X is closed if and only if the limits in X of every convergent net $\{x_i\}_{i \in I}$ of points of $\mathcal{C}$ lie in $\mathcal{C}$.*

3. *A map $f : X \to Y$ is continuous if and only if for every net $\{x_i\}_{i \in I}$ in X converging to $x \in X$, the net $\{f(x_i)\}_{i \in I}$ in Y converges to $f(x)$.*

Moreover, we are going to work essentially with the following two examples of topological spaces, for which the definition of convergence of nets admits a handy characterization.

Example 2.1.3.

1. In $\mathbb{R}$ together with the standard topology, a net $\{x_i\}_{i\in I}$ converges to $x \in \mathbb{R}$ if and only for every $\epsilon > 0$, there exists $i_0 \in I$ such that for every $i \geq i_0$, we have $|x_i - x| < \epsilon$.

   The usual properties of sequences in $\mathbb{R}$ extend to nets. For example:

   a) If $x_i \to x$ and $y_i \to y$, then $x_i + y_i \to x + y$ and $x_i - y_i \to x - y$.

   b) If $x_i \to x$ and $y_i \to y$ with $y, y_i \neq 0$ for every $i \in I$, then $\dfrac{x_i}{y_i} \to \dfrac{x}{y}$.

   c) If $x_i \to x$, $y_i \to y$ and $x_i \leq y_i$ for every $i \in I$, then $x \leq y$.

2. Let X be a set, Y a topological space and $\mathcal{M}$ a set of maps $f : X \to Y$. Then $\mathcal{M}$ can be seen as a subset of Y^X, the cartesian product of $|X|$ copies of Y with the product topology. The induced topology in $\mathcal{M}$ under this identification is the so-called *pointwise convergence topology.*

   The name comes from the fact that a net $\{f_i\}_{i\in I}$ in $\mathcal{M}$ converges to $f \in \mathcal{M}$ if and only if for every $x \in X$, the net $\{f_i(x)\}_{i\in I}$ converges to $f(x)$ in Y.

$\square$

Let for the moment $\mathbb{P}_{\mathrm{Mat}}(R)$ denote the set of Sylvester matrix rank functions on the ring R. As in Example 2.1.3(2.), $\mathbb{P}_{\mathrm{Mat}}(R)$ can be identified with a subset of $\mathbb{R}^{\mathrm{Mat}(R)}$ and hence endowed with the pointwise convergence topology. In particular, given any net $\{\mathrm{rk}_i\}_{i\in I}$ in $\mathbb{P}_{\mathrm{Mat}}(R)$ and any point in $\mathbb{R}^{\mathrm{Mat}(R)}$, seen as a map $f : \mathrm{Mat}(R) \to \mathbb{R}$, we have that $\mathrm{rk}_i \to f$ in $\mathbb{R}^{\mathrm{Mat}(R)}$ if and only if for every matrix $A \in \mathrm{Mat}(R)$, $\mathrm{rk}_i(A) \to f(A)$ in $\mathbb{R}$.

Therefore, every claim we do about $\mathbb{P}_{\mathrm{Mat}}(R)$ reduces to checking the convergence of the appropriate nets in $\mathbb{R}$. And note also that, as a product of Hausdorff spaces, $\mathbb{R}^{\mathrm{Mat}(R)}$ is again Hausdorff, and hence the limit of a convergent net is unique (Proposition 2.1.2(1.)).

Proposition 2.1.4. $\mathbb{P}_{\mathrm{Mat}}(R)$ *is a compact convex subset of* $\mathbb{R}^{\mathrm{Mat}(R)}$.

Proof. One can show that if rk_1 and rk_2 are Sylvester matrix rank functions on R then, for every $0 \leq \lambda \leq 1$, the map $\lambda\,\mathrm{rk}_1 + (1-\lambda)\,\mathrm{rk}_2$ is again a Sylvester matrix rank function on R, so $\mathbb{P}_{\mathrm{Mat}}(R)$ is a convex set.

Observe now that if $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(R)$ and $n(A)$ denotes the number of rows of a given matrix A over R, then $0 \leq \mathrm{rk}(A) \leq n(A)$. Thus, we can think of $\mathbb{P}_{\mathrm{Mat}}(R)$ as a subset of $\prod_A [0, n(A)]$, which is compact in $\mathbb{R}^{\mathrm{Mat}(R)}$ by Tychonoff's theorem . Since closed subsets of compact sets are also compact, we just need to check that $\mathbb{P}_{\mathrm{Mat}}(R)$ is closed, and by Proposition 2.1.2(2.) this is equivalent to show that given a net $\{\mathrm{rk}_i\}_{i\in I}$ in $\mathbb{P}_{\mathrm{Mat}}(R)$ that converges in $\mathbb{R}^{\mathrm{Mat}(R)}$ to a map $f : \mathrm{Mat}(R) \to \mathbb{R}$, we have $f \in \mathbb{P}_{\mathrm{Mat}}(R)$. Indeed,

(SMat1): Since $\mathrm{rk}_i(1) = 1$ for every i and $\mathrm{rk}_i(1) \to f(1)$ in $\mathbb{R}$, then necessarily $f(1) = 1$. Similarly $f(0) = 0$ for every 0 matrix.

(SMat2): Given matrices A and B that can be multiplied, we have in $\mathbb{R}$ that $\mathrm{rk}_i(AB) \to f(AB)$ and $\mathrm{rk}_i(A) \to f(A)$. Since $\mathrm{rk}_i(AB) \le \mathrm{rk}_i(A)$ for every $i \in I$, we have by Example 2.1.3(1.c) that $f(AB) \le f(A)$. Similarly $f(AB) \le f(B)$.

(SMat3): Let A, B be two matrices over R. We have that $\mathrm{rk}_i(A) \to f(A)$, $\mathrm{rk}_i(B) \to f(B)$ and $\mathrm{rk}_i(A \oplus B) \to f(A \oplus B)$ in $\mathbb{R}$. From the first two and Example 2.1.3(1.a) we obtain that the net $\{\mathrm{rk}_i(A) + \mathrm{rk}_i(B)\}_{i \in I}$ converges to $f(A) + f(B)$. Since $\mathrm{rk}_i(A \oplus B) = \mathrm{rk}_i(A) + \mathrm{rk}_i(B)$ for every $i \in I$, we have by uniqueness of the limit that $f(A \oplus B) = f(A) + f(B)$.

(SMat4:) Let A, B, C be matrices of appropriate sizes. As before, we have that $\mathrm{rk}_i(A \oplus B) \to f(A) + f(B)$, and $\mathrm{rk}_i \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \to f \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ in $\mathbb{R}$. Since $\mathrm{rk}_i \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \ge \mathrm{rk}_i(A \oplus B)$ for every $i \in I$, again by Example 2.1.3(1.c) we deduce that

$$f \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \ge f(A) + f(B)$$

Recall that (SMat2) and (SMat1) are enough to ensure the non-negativity of f, so f is actually a Sylvester matrix rank function and $\mathbb{P}_{\mathrm{Mat}}(R)$ is closed. This finishes the proof. □

To describe the space $\mathbb{P}_{\mathrm{Mat}}(R)$, we also need the following terminology.

Definition 2.1.5.

- A Sylvester matrix rank function $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(R)$ is called *extreme* or *an extreme point of* $\mathbb{P}_{\mathrm{Mat}}(R)$ if it admits no non-trivial expression as a convex combination of two different elements in $\mathbb{P}_{\mathrm{Mat}}(R)$.

- If X and Y are convex sets in the $\mathbb{R}$-vector spaces E_1 and E_2, respectively, then a map $f : X \to Y$ is *convex-linear* if it preserves finite convex combinations.

Although we will not discuss this in detail, $\mathbb{R}^{\mathrm{Mat}(R)}$ is not only Hausdorff, but a locally convex topological vector space. As a consequence, Krein-Milman theorem states that any compact convex subset $\mathcal{K}$ of $\mathbb{R}^{\mathrm{Mat}(R)}$ equals the closure of the convex hull of its extreme points. Moreover, we can allow in such subsets (countably) infinite convex combinations of points (cf. [Goo91, Proposition A.7]), i.e., if we have an infinite family of points $\{x_i\}_{i \in \mathbb{N}}$ in $\mathcal{K}$ and non-negative real numbers $\{\lambda_i\}_{i \in \mathbb{N}}$ adding up to 1, then the partial sums $\sum_{i=1}^n \lambda_i x_i$ converge to a unique point x in $\mathcal{K}$ and we set $x = \sum \lambda_i x_i$. In the proof of [Goo91, Proposition A.7] it is shown that if $\mu_i = \sum_{j=1}^i \lambda_j$, then $\{\frac{1}{\mu_i} \sum_{j=1}^n \lambda_j x_j\}_{i \in \mathbb{N}}$ is a sequence in $\mathcal{K}$ that converges to x. As a consequence, convex-linear homeomorphisms preserve these infinite convex combinations.

This justifies our attempt to identify the extreme points in $\mathbb{P}_{\mathrm{Mat}}(R)$. Furthermore, in the families of rings we are going to deal with, we show that any Sylvester rank function is uniquely a (possibly infinite) convex combination of those extreme points.

Notice that we can analogously define $\mathbb{P}_{\mathrm{Mod}}(R)$ as the set of all Sylvester module rank functions on R and identify it with a compact convex subset of $\mathbb{R}^{\text{FP-Mod}}$, where

FP-Mod denotes the set of all finitely presented left R-modules up to R-isomorphism. We show in the next proposition that the bijection in Proposition 1.2.8 defines actually a convex-linear homeomorphism between $\mathbb{P}_{\mathrm{Mat}}(R)$ and $\mathbb{P}_{\mathrm{Mod}}(R)$.

Proposition 2.1.6. *The bijection presented in Proposition 1.2.8 defines a convex-linear homeomorphism between $\mathbb{P}_{\mathrm{Mat}}(R)$ and $\mathbb{P}_{\mathrm{Mod}}(R)$. In particular, it preserves the extreme points.*

Proof. Recall that we associate to a Sylvester matrix rank function rk the Sylvester module rank function dim defined, on a finitely presented left R-module $M \cong R^m/R^nA$, by

$$\dim(M) = m - \mathrm{rk}(A).$$

To see that the defined map is continuous, we need to check (Proposition 2.1.2(3.)) that for every net $\{\mathrm{rk}_i\}_{i\in I}$ in $\mathbb{P}_{\mathrm{Mat}}(R)$ converging to a rank $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(R)$, the net of associated Sylvester module rank functions $\{\dim_i\}_{i\in I}$ in $\mathbb{P}_{\mathrm{Mod}}(R)$ converges to the Sylvester module rank function dim associated to rk. Indeed, by definition we have that for every $n \times m$ matrix A over R, $\{\mathrm{rk}_i(A)\}_{i\in I}$ converges to $\mathrm{rk}(A)$ in $\mathbb{R}$, i.e., the net $\{m - \dim_i(R^m/R^nA)\}_{i\in I}$ converges to $m - \dim(R^m/R^nA)$. Therefore, by Example 2.1.3(1.a), $\{\dim_i(R^m/R^nA)\}_{i\in I}$ converges to $\{\dim(R^m/R^nA)\}$. Since this holds for every finitely presented module, this means precisely that $\{\dim_i\}_{i\in I}$ converges to dim in $\mathbb{P}_{\mathrm{Mod}}(R)$.

To see that the map is convex-linear, note that given a convex combination $\mathrm{rk} = \sum_{i=1}^n \lambda_i \,\mathrm{rk}_i$ of elements in $\mathbb{P}_{\mathrm{Mat}}(R)$, and if dim, $\dim_i$ denote the associated Sylvester module rank functions, then for any finitely presented module $M \cong R^m/R^nA$,

$$\begin{aligned}\dim(M) &= m - \mathrm{rk}(A) = m - \sum_{i=1}^n \lambda_i \,\mathrm{rk}_i(A) = \\ &= \sum_{i=1}^n \lambda_i(m - \mathrm{rk}_i(A)) = \sum_{i=1}^n \lambda_i \dim_i(M),\end{aligned}$$

i.e., $\dim = \sum_{i=1}^n \lambda_i \dim_i$.

Analogously, the inverse is also convex-linear and continuous, and thus this yields a convex-linear homeomorphism between the spaces. □

For this reason, we do not make in the following any structural distinction between $\mathbb{P}_{\mathrm{Mat}}(R)$ and $\mathbb{P}_{\mathrm{Mod}}(R)$. From now on, we reserve the notation $\mathbb{P}(R)$ to refer to $\mathbb{P}_{\mathrm{Mat}}(R)$, and in view of Proposition 2.1.6 we sometimes prove claims about $\mathbb{P}(R)$ by showing the corresponding result in $\mathbb{P}_{\mathrm{Mod}}(R)$.

Now that we have properly defined $\mathbb{P}(R)$, the next step is to produce Sylvester rank functions. Defining one from scratch is not easy in general, but we have already noticed in Chapter 1 before Definition 1.4.5 that if we have a ring homomorphism $\varphi : R \to S$ and a rank function rk_S on S, then we can pull back rk_S to a rank function $\mathrm{rk}_R = \varphi^\sharp(\mathrm{rk}_S) := \mathrm{rk}_S \circ \varphi$. Therefore, we have the following.

Proposition 2.1.7. *Every ring homomorphism $\varphi : R \to S$ induces a convex-linear continuous map $\varphi^\sharp : \mathbb{P}(S) \to \mathbb{P}(R)$. Moreover, if φ is surjective, then $\varphi^\sharp$ is injective, and if φ is bijective, then $\varphi^\sharp$ is a homeomorphism.*

Proof. Let $\{\mathrm{rk}'_i\}_{i\in I}$ be a net in $\mathbb{P}(S)$ converging to $\mathrm{rk}' \in \mathbb{P}(S)$, and let A be any matrix over R. From the convergence of $\{\mathrm{rk}'_i\}_{i\in I}$ we obtain in particular that

$$\mathrm{rk}'_i(\varphi(A)) \longrightarrow \mathrm{rk}'(\varphi(A))$$

But this means precisely that $\{\varphi^\sharp(\mathrm{rk}'_i)\}$ converges to $\varphi^\sharp(\mathrm{rk}')$ in $\mathbb{P}(R)$, and thus $\varphi^\sharp$ is continuous. Moreover, if we have a convex combination $\mathrm{rk}_S = \sum_{i=1}^n \lambda_i \,\mathrm{rk}_i$, then by definition, for every matrix A over R we have

$$\varphi^\sharp(\mathrm{rk}_S)(A) = \mathrm{rk}_S(\varphi(A)) = \sum_{i=1}^n \lambda_i \,\mathrm{rk}_i(\varphi(A)) = \sum_{i=1}^n \lambda_i \varphi^\sharp(\mathrm{rk}_i)(A),$$

i.e., $\varphi^\sharp(\mathrm{rk}_S) = \sum_{i=1}^n \lambda_i \varphi^\sharp(\mathrm{rk}_i)$, what proves convex-linearity.

Now, if φ is surjective and $\varphi^\sharp(\mathrm{rk}'_1) = \varphi^\sharp(\mathrm{rk}'_2)$, then since any matrix A over S can be written as $\varphi(B)$ for some matrix B over R, we deduce $\mathrm{rk}'_1(A) = \mathrm{rk}'_2(A)$, and therefore $\mathrm{rk}'_1 = \mathrm{rk}'_2$. And finally, if φ is bijective, then $\varphi^\sharp$ is a homeomorphism with inverse $(\varphi^{-1})^\sharp$. □

Hanfeng Li generalized the result for surjective maps to a broader family of ring homomorphisms, called epic homomorphisms. We give now the definition and we develop further the topic in Chapter 3 and Chapter 4 after introducing epic division and $*$-regular rings.

Definition 2.1.8. A ring homomorphism $\varphi : R \to S$ is *epic* if it is right cancellable, i.e., for any two ring homomorphisms $\psi_1, \psi_2 : S \to Q$ such that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$, we have $\psi_1 = \psi_2$.

For instance, surjective homomorphisms are epic, and for a non-surjective example, the inclusion map $R \to T^{-1}R$, where $T^{-1}R$ stands for the left Ore localization of R with respect to a multiplicative left Ore set T of non-zero-divisors (see Section 3.1), is also epic. This is clear because if two homomorphisms ψ_1, ψ_2 from $T^{-1}R$ coincide on R, then necessarily one has for every element $t^{-1}r \in T^{-1}R$,

$$\psi_1(t^{-1}r) = \psi_1(t)^{-1}\psi_1(r) = \psi_2(t)^{-1}\psi_2(r) = \psi_2(t^{-1}r).$$

As a particular example, the embedding $\mathbb{Z} \to \mathbb{Q}$ is epic.

Combining Hanfeng Li's result [Li20, Theorem 8.1] with Proposition 1.4.7 we obtain the next result.

Proposition 2.1.9. *If $\varphi : R \to S$ is an epic ring homomorphism, then the induced map $\varphi^\sharp : \mathbb{P}(S) \to \mathbb{P}(R)$ is injective. In particular, if $S = T^{-1}R$ is the left Ore localization of R with respect to a multiplicative left Ore set T of non-zero-divisors, then*

$$\mathrm{im}\,\varphi^\sharp = \{\mathrm{rk} \in \mathbb{P}(R) : \mathrm{rk}(t) = 1 \textit{ for every } t \in T\}.$$

For us, among the family of rank functions that can be obtained through homomorphisms, are of particular interest the regular ranks defined in Definition 1.4.5, because we will show later that we can apply to them most of the machinery developed in Section 1.5.

Definition 2.1.10. For a ring R, we denote by $\mathbb{P}_{\mathrm{reg}}(R)$ the set of all regular rank functions on R.

As it happens with $\mathbb{P}(R)$, $\mathbb{P}_{\mathrm{reg}}(R)$ enjoys some desirable closure properties. The proof presented here is the one in [Jai19, Proposition 5.9] with minor changes.

Proposition 2.1.11. *$\mathbb{P}_{\mathrm{reg}}(R)$ is a compact convex subset of $\mathbb{P}(R)$. In particular it is closed.*

Proof. Consider the set of elements of $\mathbb{P}_{\mathrm{reg}}(R)$ indexed by some index set I. For every $i \in I$ there exists a regular ring $\mathcal{U}_i$ with a rank function rk_i' and a ring homomorphism $\varphi_i : R \to \mathcal{U}_i$ such that $\mathrm{rk}_i = \varphi_i^\sharp(\mathrm{rk}_i')$. Construct $\mathcal{U} = \prod_{i\in I} \mathcal{U}_i$ and let π_i denote the canonical projection $\pi_i : \mathcal{U} \to \mathcal{U}_i$. For every $i \in I$, we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\varphi_i"'] & \mathcal{U} \arrow[d, "\pi_i"] \\ & \mathcal{U}_i \end{tikzcd}$$

where $\varphi(r) = (\varphi_i(r))_{i\in I}$. Thus, $\mathrm{rk}_i^* = \pi_i^\sharp(\mathrm{rk}_i')$ defines a rank function on $\mathcal{U}$ such that $\varphi^\sharp(\mathrm{rk}_i^*) = \mathrm{rk}_i$. Therefore, every regular rank function on R comes from $\mathcal{U}$. Note that $\mathcal{U}$ is regular because it is a product of regular rings, and hence the convex-linear continuous map $\varphi^\sharp : \mathbb{P}(\mathcal{U}) \to \mathbb{P}(R)$ satisfies that $\mathrm{im}\,\varphi^\sharp = \mathbb{P}_{\mathrm{reg}}(R)$. Since $\varphi^\sharp$ is convex-linear, its image is a convex set, and since $\varphi^\sharp$ is continuous and $\mathbb{P}(\mathcal{U})$ is compact, then its image is also compact. As a compact set of a Hausdorff space, it is closed. $\square$

As a consequence of this proposition, we can prove the result we anticipated at the end of Section 1.5, i.e., the regularity of the transcendental extension when the base ring is regular.

Corollary 2.1.12. *Let $\mathcal{U}$ be a regular ring, τ an automorphism of $\mathcal{U}$ and rk a τ-compatible Sylvester matrix rank function on $\mathcal{U}$ with natural transcendental extension $\widetilde{\mathrm{rk}}$. Then $\widetilde{\mathrm{rk}}$ is regular both as a rank on $\mathcal{U}[t;\tau]$ and on $\mathcal{U}[t^{\pm 1};\tau]$.*

Proof. As we already observed in Chapter 1, the k^{th} extension $\widetilde{\mathrm{rk}}_k$ of rk on $\mathcal{U}[t;\tau]$ is regular since it comes from the regular ring $\mathrm{Mat}_n(\mathcal{U})$ (Example 1.3.2c)). Since $\mathbb{P}_{\mathrm{reg}}(\mathcal{U}[t;\tau])$ is closed by the previous proposition, $\widetilde{\mathrm{rk}}$ is also regular as a rank on $\mathcal{U}[t;\tau]$.

Now, let $(\mathcal{U}', \varphi, \mathrm{rk}')$ be a regular envelope of rk (see Remark 1.4.6), so that rk' is faithful and $\widetilde{\mathrm{rk}} = \varphi^\sharp(\mathrm{rk}')$. Since $\widetilde{\mathrm{rk}}(t) = 1$, we deduce that

$$1 = \widetilde{\mathrm{rk}}(t) = \mathrm{rk}'(\varphi(t)).$$

Therefore, Lemma 1.3.12 tells us that $\varphi(t)$ (and hence $\varphi(t^n)$ for every non-negative integer n) is invertible in $\mathcal{U}'$. The universal property of Ore localization (see Proposition 3.1.4 and Example 3.1.7) tells us that φ extends uniquely to a ring homomorphism

$$\tilde{\varphi} : \mathcal{U}[t^{\pm 1};\tau] \to \mathcal{U}'.$$

Since $\tilde{\varphi}^\sharp(\mathrm{rk}')$ is a rank function on $\mathcal{U}[t^{\pm 1};\tau]$ that coincides with $\widetilde{\mathrm{rk}}$ in $\mathcal{U}[t;\tau]$, we deduce that $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\mathrm{rk}')$ as a rank function on $\mathcal{U}[t^{\pm 1};\tau]$, and hence $\widetilde{\mathrm{rk}}$ is regular on $\mathcal{U}[t^{\pm 1};\tau]$. $\square$

Just as a remark, not every Sylvester matrix rank function is regular, so there are rings R for which the containment $\mathbb{P}_{\text{reg}}(R) \subseteq \mathbb{P}(R)$ is strict.

Example 2.1.13. Not every Sylvester matrix rank function is regular.

Consider the ring $R = \mathbb{Z}/4\mathbb{Z}$. Since R is an artinian ring, every finitely generated R-module M has finite length $l(M)$ (in the sense of composition series). In particular, R has finite length 2 and thus, since the length is additive in short exact sequences (cf. [GW04, Proposition 4.12]), the function dim given by $\dim(M) = \frac{l(M)}{2}$ defines a Sylvester module rank function on R. Observe in particular that if rk is the Sylvester matrix rank function associated to dim, then $\mathrm{rk}(2+4\mathbb{Z}) = \frac{1}{2}$.

We claim that rk is not regular. To see this, first observe that the characteristic of a regular ring cannot be a square. Indeed, let $\mathcal{U}$ be a regular ring and assume that $\mathrm{char}(\mathcal{U}) = n^2$. Then the element $n1$ is a non-zero element in the center of $\mathcal{U}$, but by regularity we can find $y \in \mathcal{U}$ such that $n1 = (n1)y(n1) = (n^2 1)y = 0$, a contradiction. Therefore, if $\varphi : R \to \mathcal{U}$ is a ring homomorphism from R to a (non-zero) regular ring, then since $\mathrm{char}(\mathcal{U})$ must divide $\mathrm{char}(R) = 4$, we must have $\mathrm{char}(\mathcal{U}) = 2$. But then $\varphi(2+4\mathbb{Z}) = 0$ and any rank function coming from $\mathcal{U}$ gives $2+4\mathbb{Z}$ value 0. This completes the proof of the claim.

This way of producing rank functions through length of modules also appears in Section 2.2 and Section 2.3.

□

Besides isomorphic rings, there is another instance of rings for which the spaces of rank functions are homeomorphic: Morita equivalent rings. This is proved in [Goo91, Theorem 17.14] for regular rings, and in full generality in [Li20, Remark 7.1]. Here, we give another proof of this fact for the sake of completeness and in order to state the analog of [Goo91, Proposition 16.20].

Proposition 2.1.14. *Let R, S be Morita equivalent rings. Then there exists a homeomorphism between $\mathbb{P}(R)$ and $\mathbb{P}(S)$ preserving the extreme points. Moreover, if $\iota : R \to \mathrm{Mat}_n(R)$ is the diagonal embedding, this homeomorphism is convex-linear and sends $\mathrm{rk} \in \mathbb{P}(\mathrm{Mat}_n(R))$ to the rank function $\iota^\sharp(\mathrm{rk})$ and $\mathrm{rk}' \in \mathbb{P}(R)$ to the rank function $\frac{1}{n}\mathrm{rk}'$.*

Proof. Let R-Mod (resp. S-Mod) denote the category of left R-modules (resp. S-modules), and let $F : R\text{-Mod} \to S\text{-Mod}$, $G : S\text{-Mod} \to R\text{-Mod}$ be the associated equivalence between these categories with FG and GF naturally equivalent to the corresponding identity functors. Recall that an equivalence of categories preserves direct sums, short exact sequences and finitely presented modules ([Lam99, Section 18A]).

Let dim be a module rank function on S. Since $F(R)$ is a progenerator in S-Mod ([Lam99, Remark 18.10(A)]), S is a direct summand of $F(R)^n$ for some positive integer n. Therefore, from (SMod1) and (SMod2) we obtain that $\dim(F(R)) > 0$.

By the previous remarks, the expression $\dim'(N) = \frac{\dim(F(N))}{\dim(F(R))}$ for an R-module N now defines a Sylvester module rank function on R, so we have a map $\mathbb{P}(S) \to \mathbb{P}(R)$. Similarly, we can define a map $\mathbb{P}(R) \to \mathbb{P}(S)$. Finally, by the natural equivalence, we have $FG(M) \cong M$ for every S-module M and $GF(N) \cong N$ for every R-module N. As a consequence, one can check that both maps are mutual inverses.

To check the continuity of the given map, let $\{\dim_i\}_{i\in I}$ be a net of rank functions on $\mathbb{P}(S)$ converging to $\dim \in \mathbb{P}(S)$. As we explained before, $\dim(F(R)) > 0$ and $\dim_i(F(R)) > 0$ for every $i \in I$. Since by convergence we have that for every finitely presented S-module M, $\dim_i(M) \to \dim(M)$ in $\mathbb{R}$, then in particular using Example 2.1.3(1.b), we obtain that for every finitely presented R-module N,

$$\frac{\dim_i(F(N))}{\dim_i(F(R))} \longrightarrow \frac{\dim(F(N))}{\dim(F(R))} \text{ in } \mathbb{R},$$

what means precisely that the net $\{\dim_i'\}_{i\in I}$ converges to $\{\dim'\}$ in $\mathbb{P}(R)$. Thus, the map $\mathbb{P}(S) \to \mathbb{P}(R)$ is continuous, and similarly $\mathbb{P}(R) \to \mathbb{P}(S)$ is also continuous. Thus, they are homeomorphisms.

Finally, they preserve extreme points. For instance, if we have a convex combination $\dim = \lambda \dim_1 + (1-\lambda)\dim_2$ in $\mathbb{P}(S)$, where $1 > \lambda > 0$, then we obtain a linear combination

$$\dim' = \frac{\lambda \dim_1(F(R))}{\dim(F(R))}\dim_1' + \frac{(1-\lambda)\dim_2(F(R))}{\dim(F(R))}\dim_2'$$

which is convex, since the coefficients are positive and add up to one.

For the particular case of $S = \mathrm{Mat}_n(R)$, the equivalences of categories are defined on objects (cf. [Lam99, Theorem 17.20]) by

$$\begin{array}{cccc} F: & R\text{-Mod} & \to & S\text{-Mod} \\ & P & \mapsto & P^{n\times 1} \end{array}, \qquad \begin{array}{cccc} G: & S\text{-Mod} & \to & R\text{-Mod} \\ & Q & \mapsto & E_{11}Q \end{array}$$

where E_{11} is the $n\times n$ matrix having 1 in the upper left corner and zeros everywhere else. Observe that $E_{11}Q \cong R^{1\times n}\otimes_S Q$ as R-modules and $P^{n\times 1} \cong R^{n\times 1}\otimes_R P$ as S-modules. Here, we use the notation $R^{1\times n}$ and $R^{n\times 1}$ to make explicit that we are considering rows and columns, respectively, for the module operations to be defined naturally.

Let rk be a Sylvester matrix rank function on S with associated module rank function dim. Since $F(R)^n \cong S$, we have $\dim(F(R)) = \frac{1}{n}$. Moreover, for any $A \in \mathrm{Mat}_{k\times l}(R)$, if $M = R^l/R^kA$, there exists an isomorphism of S-modules

$$\bigoplus_{i=1}^n F(M) = \bigoplus_{i=1}^n M^{n\times 1} \cong S^l/S^kB$$

where $B = \iota(A)$. Thus, if rk' is the Sylvester matrix rank function associated to $\dim'$ as defined before, then

$$\begin{aligned} \mathrm{rk}'(A) &= l - \dim'(M) = l - \frac{\dim(M^{n\times 1})}{\dim(F(R))} = l - n\dim(M^{n\times 1}) = \\ &= l - \dim(S^l/S^kB) = \mathrm{rk}(B) = \iota^\sharp(\mathrm{rk})(A), \end{aligned}$$

so we conclude that $\mathrm{rk}' = \iota^\sharp(\mathrm{rk})$.

Conversely, if rk' is a Sylvester matrix rank function on R with associated module rank function $\dim'$, and we denote by rk and dim the corresponding rank functions on S given by the Morita equivalence, then, from the R-module isomorphisms

$$G(S) \cong R^n \quad \text{and} \quad R^{1\times n}\otimes_S S^l/S^kB \cong R^{nl}/R^{nk}B$$

where $B \in \mathrm{Mat}_{k\times l}(S)$ is considered on the right as an $nk \times nl$ matrix over R, we obtain that

$$\begin{aligned}\mathrm{rk}(B) &= l - \dim(S^l/S^kB) = l - \frac{\dim'(R^{1\times n} \otimes_S S^l/S^kB)}{\dim'(G(S))} = \\ &= l - \frac{1}{n}\dim'(R^{nl}/R^{nk}B) = l - \frac{1}{n}(nl - \mathrm{rk}'(B)) = \frac{1}{n}\mathrm{rk}'(B),\end{aligned}$$

from where $\mathrm{rk} = \frac{1}{n}\mathrm{rk}'$. Since this correspondence preserves convex combinations, this finishes the proof. □

Observe from the previous construction that, in general, the homeomorphism constructed between the spaces of rank functions of Morita equivalent rings is not convex-linear. A counterexample for convex-linearity is given in [Goo91, Example 17.15].

We finish the section by studying some examples of rings for which we can describe completely the space of Sylvester rank functions. The first one was already mentioned at the beginning of Section 1.3.

Example 2.1.15.

1. On a division ring $\mathcal{D}$, there exists only one rank function, namely, the usual rank $\mathrm{rk}_{\mathcal{D}}$. Indeed, since $\mathcal{D}$ is regular, every rank function on $\mathcal{D}$ is determined by its values on elements, but every non-zero element in $\mathcal{D}$ must have rank 1 because it is invertible. Therefore, $\mathbb{P}(\mathcal{D}) = \{\mathrm{rk}_{\mathcal{D}}\}$.

2. By the previous example and Proposition 2.1.14, for a matrix ring $R = \mathrm{Mat}_n(\mathcal{D})$ over a division ring $\mathcal{D}$, we have $\mathbb{P}(R) = \{\frac{1}{n}\mathrm{rk}_{\mathcal{D}}\}$.

□

In the following, we relate the space of rank functions on a finite cartesian product of rings with the space on every factor. Together with the previous examples, this gives a complete description of the space of rank functions on a semisimple artinian ring. The example of finite cartesian products of regular rings (and in particular semisimple artinian rings) was already studied in [Goo91, Theorem 16.5 & Corollary 16.6]. The next observation is in order to make precise the statement of the proposition.

Remark 2.1.16. Let $R = R_1 \times \cdots \times R_n$ and let $\pi_i : R \to R_i$ denote the natural projections. We say that the rank $\mathrm{rk} \in \mathbb{P}(R)$ can be uniquely expressed as a convex combination of ranks on R_i if there exist uniquely determined non-negative coefficients $\lambda_1, \dots, \lambda_n$ with $\sum \lambda_i = 1$ and, for every $\lambda_i > 0$, a uniquely determined $\mathrm{rk}_i \in \mathbb{P}(R_i)$, such that $\mathrm{rk} = \sum \lambda_i \pi_i^\sharp(\mathrm{rk}_i)$.

The assumption that rk_i exists and it is uniquely determined only for $\lambda_i > 0$ is needed (cf. [Goo91, Theorem 16.5]) to address properly the cases in which some R_i does not admit Sylvester rank functions or that $\lambda_i = 0$ and R_i admits more than one Sylvester rank function (in which case the expression would not be unique). □

Proposition 2.1.17. *Let R_1, R_2 be rings. Any Sylvester rank function on $R = R_1 \times R_2$ is a uniquely determined convex combination of Sylvester rank functions on R_1 and R_2.*

In particular, the set of extreme points on $\mathbb{P}(R)$ is the disjoint union of the sets of extreme points of $\mathbb{P}(R_1)$ and $\mathbb{P}(R_2)$.

Proof. Let $\pi_i : R \to R_i$ be the natural projections. Since π_i is a surjective ring homomorphism, the map $\pi_i^\sharp : \mathbb{P}(R_i) \to \mathbb{P}(R)$ is injective. Consider also the natural additive maps $\iota_i : R_i \to R$.

Observe that if $A \in \mathrm{Mat}_{n\times m}(R)$, then $A = A_1 + A_2$ where $A_1 = \iota_1\pi_1(A)$, $A_2 = \iota_2\pi_2(A)$. Moreover, if $I_{n,1} \in \mathrm{Mat}_n(R)$, $I_{m,2} \in \mathrm{Mat}_m(R)$ denote the diagonal matrices whose entries are all equal to $(1,0)$ and $(0,1)$, respectively, we have

$$I_{n,1}A_1 = A_1, \quad I_{n,1}A_2 = 0, \quad A_1 I_{m,2} = 0, \quad A_2 I_{m,2} = A_2.$$

Thus, Lemma 1.2.5 tells us that for any rank function $\mathrm{rk} \in \mathbb{P}(R)$, we have

$$\mathrm{rk}(A) = \mathrm{rk}(A_1) + \mathrm{rk}(A_2).$$

In particular, we obtain that $1 = \mathrm{rk}((1,0)) + \mathrm{rk}((0,1))$. Now, if $\mathrm{rk}((1,0)) = 0$, then $\mathrm{rk}(A_1) = \mathrm{rk}(I_{n,1}A_1) \leq \mathrm{rk}(I_{n,1}) = n\,\mathrm{rk}((1,0)) = 0$, and if $\mathrm{rk}((1,0)) > 0$, then the expression $\mathrm{rk}_1 = \frac{1}{\mathrm{rk}((1,0))}\mathrm{rk}\circ\iota_1$ defines a Sylvester matrix rank function on R_1. Similarly, if $\mathrm{rk}((0,1)) = 0$ then $\mathrm{rk}(A_2) = 0$, and we can define a rank function rk_2 on R_2 if $\mathrm{rk}((0,1)) > 0$. One can check that

$$\mathrm{rk} = \mathrm{rk}((1,0))\pi_1^\sharp(\mathrm{rk}_1) + \mathrm{rk}((0,1))\pi_2^\sharp(\mathrm{rk}_2),$$

where we understand that rk_i is considered only when the coefficient is non-zero. Moreover, if we had another expression $\mathrm{rk} = \lambda\pi_1^\sharp(\mathrm{rk}_1') + (1-\lambda)\pi_2^\sharp(\mathrm{rk}_2')$ for some $\mathrm{rk}_i' \in \mathbb{P}(R_i)$ and $0 \leq \lambda \leq 1$, then $\mathrm{rk}((1,0)) = \lambda$ and $\mathrm{rk}((0,1)) = 1-\lambda$. Hence, if $\lambda > 0$, then for every matrix B over R_1,

$$\mathrm{rk}(\iota_1(B)) = \lambda\,\mathrm{rk}_1(B) = \lambda\,\mathrm{rk}_1'(B),$$

from where $\mathrm{rk}_1 = \mathrm{rk}_1'$, and similarly, if $1-\lambda > 0$, then $\mathrm{rk}_2 = \mathrm{rk}_2'$. Thus, the combination is unique.

From the previous expression, one can also deduce that if rk is an extreme point in $\mathbb{P}(R)$, then either $\mathrm{rk}((1,0)) = 1$ and rk_1 is an extreme point in $\mathbb{P}(R_1)$ or $\mathrm{rk}((0,1)) = 1$ and rk_2 is an extreme point in $\mathbb{P}(R_2)$.

Conversely, if, for instance, rk_1 is an extreme point in $\mathbb{P}(R_1)$, then $\pi_1^\sharp(\mathrm{rk}_1)$ is a rank function on R which takes value 1 on $(1,0)$. Therefore, if we had $\pi_1^\sharp(\mathrm{rk}_1) = \lambda\,\mathrm{rk} + (1-\lambda)\mathrm{rk}'$ with $1 > \lambda > 0$ and $\mathrm{rk} \neq \mathrm{rk}'$ on $\mathbb{P}(R)$, then necessarily $\mathrm{rk}((1,0)) = \mathrm{rk}'((1,0)) = 1$ (and consequently $\mathrm{rk}((0,1)) = \mathrm{rk}'((0,1)) = 0$). Thus, reasoning as before, we can see that $\mathrm{rk}\circ\iota_1$ and $\mathrm{rk}'\circ\iota_1$ define rank functions on R_1 such that $\mathrm{rk} = \pi_1^\sharp(\mathrm{rk}\circ\iota_1)$ and $\mathrm{rk}' = \pi_1^\sharp(\mathrm{rk}'\circ\iota_1)$ (in particular, they are different) and

$$\mathrm{rk}_1 = \lambda\,\mathrm{rk}\circ\iota_1 + (1-\lambda)\mathrm{rk}'\circ\iota_1,$$

which contradicts the fact that rk_1 is extreme. This finishes the proof. □

Before moving on to the next section, we recall again that all the rings we are going to consider from now on are left noetherian, and hence the words finitely presented and finitely generated are interchangeable for left R-modules. (cf. [Rot09, Corollary 3.19]).

2.2 Left artinian primary rings

In this section we study the space of Sylvester matrix rank functions on a family of rings which is deeply related to the families in Section 2.3 and Section 2.5, namely, left artinian primary rings. More precisely, we give a description of this space for those left artinian primary rings whose Jacobson radical is generated by a central element. Throughout the section, $J(R)$ denotes the Jacobson radical of the ring R.

Following [Pie82], a ring ($\mathbb{Z}$-algebra) R is *local* if $R/J(R)$ is a division ring, or equivalently, if the set of all non-units in R form a (two-sided) ideal, which is necessarily $J(R)$ (cf. the proof of [Pie82, Proposition 5.2]). A ring R is *primary* if $R/J(R)$ is simple.

When R is left (or right) artinian, we can reduce the study of the space of rank functions on the latter family of rings to the study of local rings through Proposition 2.1.14 and the following result ([Pie82, Proposition 6.5a]).

Proposition 2.2.1. *If R is a left artinian primary ring, then there exist a unique s and a unique (up to isomorphism) left artinian local ring S, such that $R \cong \mathrm{Mat}_s(S)$.*

The following example of local artinian ring serves as a motivation for the general treatment.

2.2.1 The case of $K[t]/(t^n)$

If K is a commutative field and n is a positive integer, then the ring $R = K[t]/(t^n)$ is an example of local artinian ring. As a consequence of the structure of modules on $K[t]$, every finitely generated R-module can be expressed as a direct sum of the indecomposable R-modules $K[t]/(t^i) \cong R/(t^i + (t^n))$, $1 \leq i \leq n$. Thus, from (SMod2), any Sylvester module rank function on R is determined by its values on these modules, or equivalently, any Sylvester matrix rank function is determined by its values on the elements $t^i + (t^n)$.

We can define n Sylvester matrix rank functions $\mathrm{rk}_1, \dots, \mathrm{rk}_n$ on R through the canonical homomorphisms $R \to \mathrm{End}_K(K[t]/(t^k)) \cong \mathrm{Mat}_k(K)$ with $p + (t^n) \mapsto \phi_k^p \mapsto A_p$, where ϕ_k^p is the endomorphism given by right multiplication by $p + (t^k)$ and A_p is its associated matrix with respect to the canonical basis in $K[t]/(t^k)$ (here, we consider endomorphisms acting on the right, so both are ring homomorphisms). If rk_K denotes the usual rank function on K, then the unique rank function on $\mathrm{Mat}_k(K)$ is $\frac{1}{k}\mathrm{rk}_K$, and when we pull it back to R we obtain a regular rank function rk_k satisfying

$$\mathrm{rk}_k(t^i + (t^n)) = \begin{cases} \frac{k-i}{k} & \text{if } i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Moreover, any other rank function on R is a convex combination of the above ranks.

Proposition 2.2.2. *Let K be a field, n a positive integer and set $R = K[t]/(t^n)$. There exist exactly n extreme points in $\mathbb{P}(R)$, which are the Sylvester matrix rank functions $\mathrm{rk}_1, \dots, \mathrm{rk}_n$, and any other rank function can be uniquely expressed as a convex combination of the previous ones. As a consequence, $\mathbb{P}(R) = \mathbb{P}_{\mathrm{reg}}(R)$.*

Proof. Let rk be any rank function on R, and consider the system

$$\mathrm{rk} = \sum_{k=1}^{n} c_k \,\mathrm{rk}_k, \;\; c_k \geq 0, \;\; \sum_{k=1}^{n} c_k = 1.$$

As observed previously, any rank function on R is determined by its values on $t^i + (t^n)$, $1 \leq i \leq n-1$. Therefore, for this system to have a solution it is enough to find non-negative c_k satysfying

$$\mathrm{rk}(t^i + (t^n)) = \sum_{k=1}^{n} c_k \,\mathrm{rk}_k(t^i + (t^n)) = \sum_{k=i}^{n} c_k \frac{k-i}{k}$$

for any $0 \leq i \leq n-1$, since the equality for $i = 0$ already encodes that the coefficients add up to 1. Setting $b_k = \mathrm{rk}(t^k + (t^n)) - \mathrm{rk}(t^{k+1} + (t^n))$ for $0 \leq k \leq n-1$, we obtain that $b_k = \sum_{j=k+1}^{n} \frac{1}{j} c_j$, and so the only possible solution is given by

$$c_k = k(b_{k-1} - b_k), \;\; k = 1, \ldots, n-1, \quad c_n = n b_{n-1}$$

Finally, every $c_k \geq 0$ by Lemma 1.2.4 and $\sum_{k=1}^{n} c_k = \sum_{k=0}^{n-1} b_k = \mathrm{rk}(1 + (t^n)) = 1$. Since the solution is unique for every rank, $\mathrm{rk}_1, \ldots, \mathrm{rk}_n$ are the only rank functions that cannot be expressed as a convex combination of two different ranks and hence they are the extreme points in $\mathbb{P}(R)$.

The last assertion follows from the regularity of rk_k and the convexity of $\mathbb{P}_{\mathrm{reg}}(R)$. $\square$

While the point of view of matrices is usually easier to understand at first, working with Sylvester module rank functions allows us to use properties that do not have an analog for matrices. In this sense, observe that the associated extreme Sylvester module rank functions $\dim_1, \ldots, \dim_n$ are determined by

$$\dim_k(R/(t^i + (t^n))) = \begin{cases} \frac{i}{k} & \text{if } i \leq k \\ 1 & \text{otherwise} \end{cases}$$

2.2.2 The general case

Let us now consider any left artinian local ring R. This implies in particular that $J(R)$ is nilpotent (cf. [Pie82, Proposition 4.4] or [GW04, Theorem 4.15]). We show that if we further assume that $J(R)$ is generated by a central element, then essentially the same classification presented above still holds.

Thus, assume that $c \in Z(R)$ is such that $J(R) = (c)$, and let n be the smallest positive integer such that $c^n = 0$. Mirroring the previous example we are going to show that every rank function on R is determined by its values on $c, \ldots, c^{n-1}$ and that the expressions

$$\dim_k(R/(c^i)) = \begin{cases} \frac{i}{k} & \text{if } i \leq k \\ 1 & \text{otherwise} \end{cases}$$

can be uniquely extended to Sylvester module rank functions $\dim_1, \dots, \dim_n$ on R that turn out to be the extreme points in $\mathbb{P}(R)$.

Let us first study the structure of modules over this local ring. Since R is local, an element x is either a unit or belongs to $J(R)$. Let x be a non-zero element of $J(R)$ and m the positive integer such that $x \in J(R)^m \setminus J(R)^{m+1}$. Since $c \in Z(R)$, then $J(R)^m = (c^m)$ and therefore $x = c^m u$ for some unit $u \in R$.

Now, take $A \in \mathrm{Mat}_{k\times l}(R)$ and express every element of A as $a_{ij} = c^{m_{ij}} u_{ij}$ where u_{ij} is a unit (here, $m_{ij} = 0$ if a_{ij} is already a unit, and $m_{ij} = n$ if $a_{ij} = 0$). Multiplying by invertible matrices we can assume that $m_{11} = \min\{m_{ij}\}$ and $a_{11} = c^{m_{11}}$. Thus, if $r_{ij} = m_{ij} - m_{11}$,

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ -c^{r_{21}}u_{21} & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ -c^{r_{k1}}u_{k1} & 0 & \dots & 1 \end{pmatrix} A \begin{pmatrix} 1 & -c^{r_{12}}u_{12} & \dots & -c^{r_{1l}}u_{1l} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} c^{m_{11}} & 0 \\ 0 & B \end{pmatrix}$$

for some $(k-1)\times(l-1)$ matrix B. Using induction we see that A is equivalent to a matrix of the form $\begin{pmatrix} D & 0 \end{pmatrix}$ or $\begin{pmatrix} D \\ 0 \end{pmatrix}$ where D is a diagonal matrix whose entries are either powers c^i for some $0 \le i \le n-1$ or zero.

This expression for a matrix implies that every finitely presented left R-module M can be written in the form

$$M \cong R^m \oplus \bigoplus_{i=1}^{r} R/(c^{m_i}).$$

Moreover, since $c \in Z(R)$ and R is local, for each $0 \le i \le n$ we have that $R/(c^i)$ is an R-bimodule (with the usual operations) of length i. Indeed, under the previous hypothesis one can show that $J(R)^{k-1}/J(R)^k \cong R/J(R)$ for every $1 \le k \le n$, from where it is a simple left R-module, and hence the chain

$$0 < J(R)^{i-1}/J(R)^i < \dots < J(R)/J(R)^i < R/J(R)^i$$

is a composition series for $R/J(R)^i = R/(c^i)$ of length i (in particular, $l({}_RR) = n$).

In addition, we have R-bimodule isomorphisms (cf. [Rot09, Proposition 2.68])

$$R/(c^i) \otimes_R R/(c^j) \cong R/(c^{\min\{i,j\}}).$$

Using this fact, one can show that the previous expression for the R-module M is unique (up to reorganization of factors) by tensoring with $R/(c^i)$ for every $i = 1, \dots, n$ and comparing lengths.

Proposition 2.2.3. *Let R be a left artinian local ring, and assume that there exists an element $c \in Z(R)$ with order of nilpotency n such that $J(R) = (c)$. Then any rank function on R is determined by its values on c^i for $1 \le i \le n-1$, and the expressions $\dim_1, \dots, \dim_n$ extend uniquely to Sylvester module rank functions on R that are precisely the extreme points in $\mathbb{P}(R)$. Any other rank function can be uniquely expressed as a convex combination of the previous ones.*

Proof. The previous expression for a finitely presented left R-module, together with (SMod2), shows that there exists only one way to extend $\dim_k$ for an arbitrary finitely presented module. More precisely, if we split the decomposition of M as

$$M = R^m \oplus \bigoplus_{\substack{m_i > k \\ i=1,\dots,r_1}} R/(c^{m_i}) \oplus \bigoplus_{\substack{n_j \leq k \\ j=1,\dots,r_2}} R/(c^{n_j})$$

then $\dim_k(M) = m + r_1 + \sum_j \frac{n_j}{k}$. Thus, observe that

$$\dim_k(M) = \frac{l(R/(c^k) \otimes_R M)}{k}$$

where $l(N)$ stands for the length of N. Since the tensor commutes with direct sums and it is right exact, and the length of a module is additive on short exact sequences, we deduce that $\dim_k$ satisfies (SMod1)-(SMod3).

Since any finitely presented module can be written in the above form, we deduce that any Sylvester matrix rank function is determined by its values on $c, \dots, c^{n-1}$. Thus, noting that the associated matrix rank functions are the analogues to the extreme ranks on $K[t]/(t^n)$, the same argument of Proposition 2.2.2 shows that these are the extreme points in $\mathbb{P}(R)$. $\square$

Corollary 2.2.4. *Let R be a left artinian primary ring, and assume that there exists an element $c \in Z(R)$ with order of nilpotency n such that $J(R) = (c)$. Then any rank function on R is determined by its values on c^i for $1 \leq i \leq n-1$, and the extreme points in $\mathbb{P}(R)$ are the Sylvester matrix rank functions $\mathrm{rk}_1, \dots, \mathrm{rk}_n$ defined by*

$$\mathrm{rk}_k(c^i) = \begin{cases} \frac{k-i}{k} & \textit{if } i \leq k \\ 0 & \textit{otherwise} \end{cases}$$

Any other rank function can be uniquely expressed as a convex combination of them.

Proof. By Proposition 2.2.1, there exist a left artinian local ring S, a positive integer s and a ring isomorphism $\varphi : R \to \mathrm{Mat}_s(S)$. Notice that, since φ is an isomorphism, we have that $\varphi(J(R)) = J(\mathrm{Mat}_s(S)) = \mathrm{Mat}_s(J(S))$ and it is generated by $\varphi(c)$.

Now, c is a central element, and hence $\varphi(c) \in Z(\mathrm{Mat}_s(S))$, from where necessarily $\varphi(c) = \mathrm{Diag}_s(d, \dots, d)$ for some $d \in J(S) \cap Z(S)$ of order n. In addition, observe that $J(S) = (d)$.

Thus, in terms of Sylvester matrix rank functions, Proposition 2.2.3 tells us that the extreme points on $\mathbb{P}(S)$ are the ranks $\mathrm{rk}'_1, \dots, \mathrm{rk}'_n$ given by

$$\mathrm{rk}'_k(d^i) = \begin{cases} \frac{k-i}{k} & \text{if } i \leq k \\ 0 & \text{otherwise} \end{cases}$$

and, by Proposition 2.1.14, $\frac{1}{s}\mathrm{rk}'_1, \dots, \frac{1}{s}\mathrm{rk}'_n$ are the extreme points in $\mathbb{P}(\mathrm{Mat}_s(S))$. Therefore, since ring isomorphisms preserve the extreme rank functions, we obtain that $\mathrm{rk}_1 =$

$\varphi^\sharp(\frac{1}{s}\operatorname{rk}'_1), \dots, \operatorname{rk}_n = \varphi^\sharp(\frac{1}{s}\operatorname{rk}'_n)$ are the extreme points in $\mathbb{P}(R)$, and taking into account that
$$\operatorname{rk}_k(c^i) = \frac{1}{s}\operatorname{rk}'_k(\operatorname{Diag}_s(d^i, \dots, d^i)) = \operatorname{rk}'_k(d^i),$$
$\operatorname{rk}_k(c^i)$ is defined as in the statement.

To finish, we need to check that rank functions on R are determined by their values on $c, \dots, c^{n-1}$. Given the bijectivity of the maps $\mathbb{P}(R) \to \mathbb{P}(\operatorname{Mat}_s(S)) \to \mathbb{P}(S)$, if two rank functions on R coincide on powers of c, their images are rank functions on S that coincide on powers of d, and hence they are equal by Proposition 2.2.3. This finishes the proof. □

2.3 Sylvester rank functions on a Dedekind domain

This section is devoted to describing the space of rank functions $\mathbb{P}(\mathcal{O})$ defined on a Dedekind domain $\mathcal{O}$ which is not a field, since the latter case has already been treated through Section 2.1. We follow [BK00] to recall the basic properties of Dedekind domains, although we also highly recommend [Nar04].

Recall that over a Dedekind domain every non-zero prime ideal is maximal and every non-zero proper ideal can be represented uniquely as a finite product of powers of distinct prime ideals (cf. [BK00, Lemma 5.1.18 & Theorem 5.1.19]). Moreover, every finitely generated $\mathcal{O}$-module M can be expressed as follows
$$M \cong \mathcal{O}^n \oplus I \oplus \left(\bigoplus_{j=1}^m \mathcal{O}/\mathfrak{m}_j^{\alpha_j}\right)$$
where I is an ideal of $\mathcal{O}$ and the $\mathfrak{m}_j$ are (non-necessarily distinct) maximal ideals (cf. [BK00, Theorem 6.3.23], where the uniqueness of such decomposition is also discussed).

The following lemma collects some other basic properties of ideals over Dedekind domains that we shall need for our purposes.

Lemma 2.3.1. *Let I be any non-zero ideal over the Dedekind domain $\mathcal{O}$. Then, the following hold.*

1. *The quotient $\mathcal{O}/I$ is an artinian principal ideal ring.*
2. *I is projective and strongly two-generated, i.e., for every non-zero $x \in I$, there exists $y \in I$ such that $I = \mathcal{O}x + \mathcal{O}y$.*
3. *For every non-zero ideal J, there exists an $\mathcal{O}$-isomorphism $I \oplus J \cong \mathcal{O} \oplus IJ$.*

Proof. The statements correspond, respectively, to [BK00, Proposition 5.1.22], [BK00, Corollary 5.1.23 & Lemma 6.1.1] and [BK00, Lemma 6.1.4]. □

Observe from Lemma 2.3.1(1) that for every maximal ideal $\mathfrak{m}$ and positive integer n, the quotient ring $\mathcal{O}/\mathfrak{m}^n$ is a local artinian ring whose unique maximal ideal $J(\mathcal{O}/\mathfrak{m}^n) =$

$\mathfrak{m}/\mathfrak{m}^n$ is (nilpotent and) principal. Let $c \in \mathfrak{m}$ be such that $c+\mathfrak{m}^n$ generates $\mathfrak{m}/\mathfrak{m}^n$ and note that, since different powers of a proper ideal in $\mathcal{O}$ are all distinct (for instance, because of the uniqueness of a primary decomposition; see also [BK00, Proposition 5.1.24]), n is precisely the order of nilpotency of this element. Then, Proposition 2.2.3 tells us that there are exactly n extreme Sylvester matrix rank functions $\mathrm{rk}_{\mathcal{O}/\mathfrak{m}^n,1}, \dots, \mathrm{rk}_{\mathcal{O}/\mathfrak{m}^n,n}$ in $\mathbb{P}(\mathcal{O}/\mathfrak{m}^n)$, which are determined by

$$\mathrm{rk}_{\mathcal{O}/\mathfrak{m}^n,k}(c^i+\mathfrak{m}^n) = \begin{cases} \frac{k-i}{k} & \text{if } i \le k \\ 0 & \text{otherwise} \end{cases}$$

Hence, we can define Sylvester matrix rank functions on $\mathcal{O}$ through the ring homomorphisms $\pi_{\mathfrak{m},n} : \mathcal{O} \to \mathcal{O}/\mathfrak{m}^n$. In particular, for any maximal ideal $\mathfrak{m}$ and positive integer k we have a rank function $\mathrm{rk}_{\mathfrak{m},k} = \pi^\sharp_{\mathfrak{m},k}(\mathrm{rk}_{\mathcal{O}/\mathfrak{m}^k,k})$. We are going to show that the associated Sylvester module rank functions $\dim_{\mathfrak{m},k}$ satisfy, for every maximal ideal $\mathfrak{n}$ and positive integer i,

$$\dim_{\mathfrak{m},k}(\mathcal{O}/\mathfrak{n}^i) = \begin{cases} \frac{i}{k} & \text{if } \mathfrak{n} = \mathfrak{m} \text{ and } i \le k \\ 1 & \text{if } \mathfrak{n} = \mathfrak{m} \text{ and } i > k \\ 0 & \text{if } \mathfrak{n} \ne \mathfrak{m} \end{cases}$$

Assume first that $\mathfrak{n} \ne \mathfrak{m}$, take any non-zero $x \in \mathfrak{n}^i \backslash \mathfrak{m}$ and let y be as in Lemma 2.3.1(2), so that $\mathfrak{n}^i = \mathcal{O}x + \mathcal{O}y$. Then $\begin{pmatrix} x \\ y \end{pmatrix}$ is a presentation matrix for $O/\mathfrak{n}^i$, and therefore

$$\dim_{m,k}(\mathcal{O}/\mathfrak{n}^i) = 1 - \mathrm{rk}_{\mathcal{O}/\mathfrak{m}^k,k}\begin{pmatrix} x+\mathfrak{m}^k \\ y+\mathfrak{m}^k \end{pmatrix} = 0,$$

where the last equality follows because, since $x \notin \mathfrak{m}$, $x+\mathfrak{m}^k$ is invertible in $\mathcal{O}/\mathfrak{m}^k$.

Now, for every $i < k$, let x be a non-zero element of $\mathfrak{m}^k$, take y such that $\mathfrak{m}^i = \mathcal{O}x + \mathcal{O}y$ and observe that necessarily $y \in \mathfrak{m}^i \backslash \mathfrak{m}^{i+1}$. Thus, if $c \in \mathfrak{m}$ is such that $c+\mathfrak{m}^k$ generates $\mathfrak{m}/\mathfrak{m}^k$, there exists an element $r \in \mathcal{O}\backslash\mathfrak{m}$, hence a unit in $\mathcal{O}/\mathfrak{m}^k$, such that $y+\mathfrak{m}^k = (r+\mathfrak{m}^k)(c^i+\mathfrak{m}^k)$, and as before,

$$\dim_{m,k}(\mathcal{O}/\mathfrak{m}^i) = 1 - \mathrm{rk}_{\mathcal{O}/\mathfrak{m}^k,k}\begin{pmatrix} x+\mathfrak{m}^k \\ y+\mathfrak{m}^k \end{pmatrix} = 1 - \mathrm{rk}_{\mathcal{O}/\mathfrak{m}^k,k}\begin{pmatrix} 0 \\ c^i+\mathfrak{m}^k \end{pmatrix} = \frac{i}{k}.$$

Finally, if $i \ge k$, the generators of $\mathfrak{m}^i$ are zero in $\mathcal{O}/\mathfrak{m}^k$, and $\dim_{m,k}(\mathcal{O}/\mathfrak{m}^i) = 1$.

As a remark here, observe that the local artinian ring $(R = \mathcal{O}/\mathfrak{m}^n, J = \mathfrak{m}/\mathfrak{m}^n)$ is complete (i.e. the natural map $R \to \lim_i R/J^i$ is an isomorphism) and separated with respect to the J-adic topology (i.e., $\bigcap_i J^i = 0$), since J is nilpotent. Thus, if we further assume that $\mathcal{O}$ contains a field k, then $k \hookrightarrow \mathcal{O} \to R$ is injective, and hence Cohen's theorem on local rings (cf. [Mat80, Theorem 60 & the proof of the subsequent Corollary 1]) tells us that there exists a subfield K of R, with $K \cong R/J \cong \mathcal{O}/\mathfrak{m}$, and a surjective ring homomorphism $K[[t]] \to R$ where t maps to the generator of J. In particular, t^n is the first power of t in the kernel of the map, and since $K[[t]]$ is a discrete valuation ring, this shows that the kernel must be precisely (t^n). Hence, we have an isomorphism $K[t]/(t^n) \cong R$, meaning in particular that in this case all rank functions on $\mathcal{O}/\mathfrak{m}^n$ are regular by Proposition 2.2.2.

We can also define the regular rank function rk_0 induced by the field of fractions $\mathcal{Q}(\mathcal{O})$. The same arguments show that the associated module rank function $\dim_0$ satisfies $\dim_0(\mathcal{O}/\mathfrak{n}^i) = 0$ for every maximal ideal $\mathfrak{n}$ and positive integer i.

The structure of finitely generated modules over $\mathcal{O}$ and the next proposition show that, in general, any Sylvester module rank function dim on $\mathcal{O}$ is determined by its values on the modules considered above.

Proposition 2.3.2. *If* dim *is a Sylvester module rank function on* $\mathcal{O}$*, then* $\dim(I) = 1$ *for every non-zero ideal* I *of* $\mathcal{O}$*.*

Proof. Let I be a non-zero ideal of $\mathcal{O}$. By using Lemma 2.3.1(3) repeatedly, we can see that for every positive integer k we have $\bigoplus_{i=1}^k I \cong \mathcal{O}^{k-1} \oplus I^k$, from where

$$\dim(I) = \frac{k-1}{k} + \frac{\dim(I^k)}{k} \geq \frac{k-1}{k}.$$

Thus, necessarily $\dim(I) \geq 1$. On the other hand, I is projective and two-generated by Lemma 2.3.1(2), and hence a direct summand of $\mathcal{O}^2$. Since an ideal of $\mathcal{O}$ cannot be free of rank 2, its complement C must be non-zero, and the structure of finitely generated modules over $\mathcal{O}$ together with the previous argument shows that $\dim(C) \geq 1$. Since $\dim(I) + \dim(C) = \dim(\mathcal{O}^2) = 2$, necessarily $\dim(I) = 1$. □

We are going to show that the previous ranks $\dim_{\mathfrak{m},k}$ and $\dim_0$ are the extreme points in $\mathbb{P}(\mathcal{O})$ by proving that any rank function dim on $\mathcal{O}$ can be uniquely written as

$$\dim = c_0 \dim_0 + \sum_{\mathfrak{m}} \sum_{k \in \mathbb{Z}^+} c_{\mathfrak{m},k} \dim_{\mathfrak{m},k}, \quad c_0, c_{\mathfrak{m},k} \geq 0, \ c_0 + \sum c_{\mathfrak{m},k} = 1,$$

where $\mathfrak{m}$ runs through all maximal ideals of $\mathcal{O}$ and k through the positive integers. As there can be an uncountable number of maximal ideals in $\mathcal{O}$, for the right hand side sum to make sense we need to show first that there are only countably many non-zero coefficients. For this purpose, note that if such an expression is to hold, then from the definition of $\dim_{\mathfrak{m},k}$ we obtain that

$$\dim(\mathcal{O}/\mathfrak{m}) = \sum_{k \geq 1} \frac{c_{\mathfrak{m},k}}{k}.$$

In particular, if $\dim(\mathcal{O}/\mathfrak{m}) = 0$, then $c_{\mathfrak{m},k} = 0$ for every k. Thus, for our goal it suffices to see that $\dim(\mathcal{O}/\mathfrak{m}) = 0$ for all but countably many maximal ideals.

Notice that the previous equality implies that if $\dim(\mathcal{O}/\mathfrak{m}) = 0$, then $\dim(\mathcal{O}/\mathfrak{m}^k) = 0$ for every k, a statement that will follow from Lemma 2.3.4 in our case but may not be true in general for a commutative ring R (From (SMod3) and the surjective homomorphism $R/\mathfrak{m}^n \to R/\mathfrak{m}$ we only deduce $\dim(R/\mathfrak{m}^n) \geq \dim(R/\mathfrak{m})$). However, we show in the following lemma that the number of pairs $(\mathfrak{m}, k)$ such that $\dim(R/\mathfrak{m}^k) > 0$ is still countable in this more general setting.

Lemma 2.3.3. *Let* dim *be a Sylvester module rank function on a commutative ring R. Then, there exist only countably many maximal ideals $\mathfrak{m}$ such that* $\dim(R/\mathfrak{m}^k) > 0$ *for some $k \geq 1$.*

Proof. Fix $k \geq 1$ and, for every $n \geq 1$, let $S_n^{(k)}$ be the collection of all maximal ideals of R with $\dim(R/\mathfrak{m}^k) > 1/n$. Suppose that there exists n such that $S_n^{(k)}$ is infinite, and take $m > n$ different maximal ideals $\{\mathfrak{m}_i\}_{i=1}^m$ in $S_n^{(k)}$. Then, using the Chinese Remainder Theorem and (SMod2),

$$\dim(R/\mathfrak{m}_1^k \cdots \mathfrak{m}_m^k) = \sum_{i=1}^m \dim(R/\mathfrak{m}_i^k) > \frac{m}{n} > 1.$$

This is a contradiction, since for every ideal I in R, we have $\dim(R/I) \leq \dim(R) = 1$ by (SMod1) and (SMod3), so $S_n^{(k)}$ must be finite for every n. Therefore, the set $S_0 = \bigcup_{k\in\mathbb{Z}^+}\bigcup_{n\in\mathbb{Z}^+} S_n^{(k)}$ is countable. □

Since the computation of coefficients is going to be very similar to that of Proposition 2.2.2, we need also the following generalization of Lemma 1.2.4.

Lemma 2.3.4. *Let* dim *be a Sylvester module rank function on $\mathcal{O}$ and let $\mathfrak{m}$ be a maximal ideal. If we set $b_{\mathfrak{m},0} = \dim(\mathcal{O}/\mathfrak{m})$ and $b_{\mathfrak{m},k} = \dim(\mathcal{O}/\mathfrak{m}^{k+1}) - \dim(\mathcal{O}/\mathfrak{m}^k)$ for $k \geq 1$, then $b_{\mathfrak{m},k} \geq b_{\mathfrak{m},k+1}$ for every $k \geq 0$.*

Proof. For any non-zero $x \in \mathfrak{m}^2$, we can find a non-zero element $y \in \mathfrak{m}\backslash\mathfrak{m}^2$ such that $\mathfrak{m} = \mathcal{O}x + \mathcal{O}y$ by Lemma 2.3.1(2). Observe that we can write then $\mathfrak{m}^{k+1} = \mathfrak{m}x^k + \mathfrak{m}^{k-1}y^2$ for every $k \geq 1$ (with $\mathfrak{m}^0 = \mathcal{O}$). One can check that the sequences

$$\mathcal{O}/\mathfrak{m} \xrightarrow{\varphi_0} \mathcal{O}/\mathfrak{m}^2 \xrightarrow{\psi_0} \mathcal{O}/\mathfrak{m} \to 0,$$

where $\varphi_0(r + \mathfrak{m}) = yr + \mathfrak{m}^2$ and $\psi_0(s + \mathfrak{m}^2) = s + \mathfrak{m}$, and

$$\mathcal{O}/\mathfrak{m}^{k+1} \xrightarrow{\varphi_k} \mathcal{O}/\mathfrak{m}^{k+2} \oplus \mathcal{O}/\mathfrak{m}^k \xrightarrow{\psi_k} \mathcal{O}/\mathfrak{m}^{k+1} \to 0,$$

where $\varphi_k(r + \mathfrak{m}^{k+1}) = (yr + \mathfrak{m}^{k+2}, r + \mathfrak{m}^k)$ and $\psi_k(s + \mathfrak{m}^{k+2}, t + \mathfrak{m}^k) = s - yt + \mathfrak{m}^{k+1}$, are all exact. In fact, every φ_k for $k \geq 0$ can be shown to be injective, but this is not needed for the proof since the result already follows from (SMod2) and (SMod3). □

We are now ready to prove the main result of this section about $\mathbb{P}(\mathcal{O})$.

Theorem 2.3.5. *The Sylvester module rank functions* $\dim_0$ *and* $\dim_{\mathfrak{m},k}$*, for every maximal ideal $\mathfrak{m}$ and $k \geq 1$, are the extreme points of $\mathbb{P}(\mathcal{O})$, and any other rank function can be uniquely expressed as a (possibly infinite) convex combination of them. In particular, if $\mathcal{O}$ contains a field, $\mathbb{P}(\mathcal{O}) = \mathbb{P}_{\mathrm{reg}}(\mathcal{O})$.*

Proof. Let dim be any Sylvester module rank function on $\mathcal{O}$. By Lemma 2.3.3, the set S_0 of all maximal ideals $\mathfrak{m}$ such that $\dim(\mathcal{O}/\mathfrak{m}^k) > 0$ for some $k \geq 1$ is countable, and as we already discussed, only coefficients $c_{\mathfrak{m},k}$ corresponding to $\mathfrak{m} \in S_0$ can be non-zero. Thus, we are going to show that there are unique non-negative numbers $c_0, c_{\mathfrak{m},k}$ summing up to 1 such that

$$\dim = c_0 \dim_0 + \sum_{\mathfrak{m}\in S_0} \sum_{k\in\mathbb{Z}^+} c_{\mathfrak{m},k} \dim_{\mathfrak{m},k} .$$

As we deduced from the structure of finitely generated modules over $\mathcal{O}$ and Proposition 2.3.2, it suffices to have equality for every module $\mathcal{O}/\mathfrak{n}^i$, where $\mathfrak{n}$ is a maximal ideal and $i \geq 1$. From the definition of $\dim_{\mathfrak{m},k}$, its only contribution for these modules is given when $\mathfrak{m} = \mathfrak{n}$. In particular, if $\mathfrak{n} \notin S_0$, the two expressions coincide on $\mathcal{O}/\mathfrak{n}^i$ for every i, and if $\mathfrak{n} \in S_0$, the coefficients $c_{\mathfrak{n},k}$ should be determined by

$$\dim(\mathcal{O}/\mathfrak{n}^i) = \sum_{k=1}^{i-1} c_{\mathfrak{n},k} + \sum_{k=i}^{\infty} c_{\mathfrak{n},k} \frac{i}{k}$$

Borrowing the notation of Lemma 2.3.4, we obtain that for every $i \geq 0$,

$$b_{\mathfrak{n},i} = \sum_{k=i+1}^{\infty} \frac{1}{k} c_{\mathfrak{n},k}.$$

Thus the only possible solution is given by the non-negative coefficients (Lemma 2.3.4)

$$c_{\mathfrak{n},k} = k(b_{\mathfrak{n},k-1} - b_{\mathfrak{n},k}), k \geq 1.$$

We still need to show that $\sum_{\mathfrak{m}\in S_0} \sum_{k=1}^{\infty} c_{\mathfrak{m},k}$ converges to a number l less than or equal to 1, and take $c_0 = 1 - l$. Notice first that $\sum_{k=0}^{n} b_{\mathfrak{m},k} = \dim(\mathcal{O}/\mathfrak{m}^{n+1})$, and hence the sequence of partial sums $\{\sum_{k=0}^{n} b_{\mathfrak{m},k}\}_n$ is monotonically increasing and bounded above by 1, so the series $\sum_{k=0}^{\infty} b_{\mathfrak{m},k}$ is convergent for every $\mathfrak{m}$. Moreover, since by (SMod3) and Lemma 2.3.4 we have $b_{\mathfrak{m},k} \geq b_{\mathfrak{m},k+1} \geq 0$ for every $k \geq 0$, Abel-Pringsheim theorem (cf. [Har08, §179]) tells us that $\lim_{k\to\infty} kb_{\mathfrak{m},k} = 0$. Therefore, from the inequalities

$$0 \leq \sum_{k=1}^{n} c_{\mathfrak{m},k} = \sum_{k=1}^{n} k(b_{\mathfrak{m},k-1} - b_{\mathfrak{m},k}) = \left[\sum_{k=0}^{n-1} b_{\mathfrak{m},k}\right] - nb_{\mathfrak{m},n} \leq \sum_{k=0}^{n-1} b_{\mathfrak{m},k} \leq 1,$$

we obtain, on the one hand, that the sequence of partial sums $\{\sum_{k=1}^{n} c_{\mathfrak{m},k}\}_n$ is also monotonically increasing and bounded above by 1, and hence that the series $\sum_{k=1}^{\infty} c_{\mathfrak{m},k}$ is also convergent. On the other hand, we also deduce from the previous discussion that $\sum_{k=1}^{\infty} c_{\mathfrak{m},k} = \sum_{k=0}^{\infty} b_{\mathfrak{m},k}$. We claim that $\sum_{\mathfrak{m}\in S_0} \sum_{k=0}^{\infty} b_{\mathfrak{m},k}$ converges to a number smaller than 1, from where the result is established. Indeed,

$$\sum_{\mathfrak{m}\in S_0} \sum_{k=0}^{\infty} b_{\mathfrak{m},k} = \sup_{\substack{B\subset S_0 \\ B \text{ finite}}} \sum_{\mathfrak{m}\in B} \sum_{k=0}^{\infty} b_{\mathfrak{m},k},$$

and hence, it suffices to prove that for every finite $B \subset S_0$, $\sum_{\mathfrak{m}\in B}\sum_k b_{\mathfrak{m},k}$ converges to a number below 1. But this follows from the Chinese Remainder Theorem, (SMod2) and (SMod3), since

$$\sum_{\mathfrak{m}\in B}\sum_{k=0}^{n} b_{\mathfrak{m},k} = \sum_{\mathfrak{m}\in B} \dim(\mathcal{O}/\mathfrak{m}^{n+1}) = \dim(\mathcal{O}/\bigcap_{\mathfrak{m}\in B}\mathfrak{m}^{n+1}) \leq 1$$

and therefore

$$\sum_{\mathfrak{m}\in B}\sum_{k=0}^{\infty} b_{\mathfrak{m},k} = \sum_{\mathfrak{m}\in B}\left[\lim_{n\to\infty}\sum_{k=0}^{n} b_{\mathfrak{m},k}\right] = \lim_{n\to\infty}\left[\sum_{\mathfrak{m}\in B}\sum_{k=0}^{n} b_{\mathfrak{m},k}\right] \leq 1,$$

where the second equality follows because we are adding a finite number of finite limits. The last assertion of the proposition is a consequence of the previous discussion regarding $\mathcal{O}/\mathfrak{m}^n$ and $K[t]/(t^n)$, and that $\mathbb{P}_{\text{reg}}(\mathcal{O})$ is closed and convex. □

2.4 Krull dimension and simple left noetherian rings

In this section we turn to the study of simple left noetherian rings. These rings appear naturally when dealing with skew Laurent polynomial rings since, for instance, for every automorphism of infinite inner order τ (see Section 2.5) of a division ring $\mathcal{D}$, the ring $\mathcal{D}[t^{\pm1};\tau]$ is simple and noetherian ([GW04, Corollary 1.15 & Theorem 1.17]). Another widely studied subfamily here are the Weyl algebras $A_n(K)$ over a field of characteristic zero ([GW04, Exercise 2G & Corollary 2.2]).

We show that on a simple left noetherian ring there exists a unique Sylvester rank function, namely, the one induced from its classical left quotient ring (i.e. the left ring of fractions with respect to the set of all non-zero-divisors). This is proved by means of induction on Krull dimension of modules, and we follow the exposition in [GW04, Chapters 15 & 16] to recall the necessary definitions and results.

Let R be a ring and let M be a left R-module. We say that the *Krull dimension* of M is -1, and we write $\mathrm{K.\dim}(M) = -1$, if and only if M is the zero module. Now, given an ordinal $\alpha \geq 0$, we write $\mathrm{K.\dim}(M) \leq \alpha$ if, for every descending chain $M_1 \geq M_2 \geq M_3 \dots$ of submodules of M, we have $\mathrm{K.\dim}(M_i/M_{i+1}) < \alpha$ for all but finitely many i. The Krull dimension of a non-zero module M is then α, denoted $\mathrm{K.\dim}(M) = \alpha$, if $\mathrm{K.\dim}(M) \leq \alpha$ and α is the least such ordinal, and we write $\mathrm{l.K.\dim}(R)$ to denote the Krull dimension of R as a left R-module.

In addition, if the module M has $\mathrm{K.\dim}(M) = \alpha \geq 0$ and all its proper factor modules have Krull dimension $< \alpha$, i.e., $\mathrm{K.\dim}(M/N) < \alpha$ for every non-zero submodule N of M, then M is called *α-critical.*

Observe, for example, that a non-zero module M has Krull dimension 0 if and only if it is artinian. Notice also that for every division ring $\mathcal{D}$ and every automorphism τ of $\mathcal{D}$, the skew Laurent polynomial ring $R = \mathcal{D}[t^{\pm1};\tau]$ is not left artinian, since we have the infinite descending chain of left ideals

$$R \supseteq R(1+t) \supseteq R(1+t)^2 \supseteq \dots,$$

and its Krull dimension is at most 1 (cf. [GW04, Theorem 15.19 & Exercise 15S]. Thus, $\mathrm{l.\,K.\,dim}(\mathcal{D}[t^{\pm 1};\tau]) = 1$ (cf. [MR01, Proposition 6.5.4 (iii)] for a more general result).

In general, an ordinal α as in the definition may not exist, so there are modules for which the Krull dimension is not defined. However, this is not the case of noetherian modules over a ring and, in particular, of finitely generated left modules over a left noetherian ring ([GW04, Lemma 15.3]).

The key point to prove the main result is the following lemma due to Stafford (cf. [Sta76, Lemma 1.4]; see also [Len00, Lemma in page 138]). He originally considered modules M with finite Krull dimension, since it turns out to give an upper bound for the minimal number of generators of M, but the same result holds without this assumption. We add a proof here for the sake of completeness, just following the lines of [GW04, Theorem 16.7]. For this purpose, recall that an R-module M is *faithful* if $\mathrm{ann}_R(M) = 0$, *fully faithful* if all its non-zero submodules are faithful, and *completely faithful* if all its non-zero factor modules are fully faithful.

Lemma 2.4.1. *Let R be a left noetherian ring and M a non-zero finitely generated completely faithful left R-module. If* $\mathrm{K.\,dim}(M) < \mathrm{l.\,K.\,dim}(R)$*, then there exists a cyclic submodule N of M such that* $\mathrm{K.\,dim}(M/N) < \mathrm{K.\,dim}(M)$.

Proof. Assume $\mathrm{K.\,dim}(M) = \alpha \geq 0$, and let $J_\alpha(M)$ denote the intersection of the kernels of all homomorphisms from M to α-critical modules (i.e., the *Krull radical of M*). Then, $J_\alpha(M)$ is a proper submodule of M, the factor module $M/J_\alpha(M)$ has Krull dimension α ([GW04, Proposition 15.11]) and hence it is fully faithful by hypothesis. Thus, [GW04, Lemma 16.4] tells us that there exists $m \in M$ such that $(Rm + J_\alpha(M))/J_\alpha(M)$ is an essential submodule of $M/J_\alpha(M)$, and this is the case if and only if $\mathrm{K.\,dim}(M/Rm) < \alpha$ by [GW04, Corollary 15.12]. □

With this, we can now state the main result.

Proposition 2.4.2. *If R is a left noetherian simple ring, then $\mathbb{P}(R) = \{\dim_{\mathcal{Q}_l(R)}\}$, where $\mathcal{Q}_l(R)$ is the classical left quotient ring of R.*

Proof. Observe that since R is left noetherian and simple, the classical left quotient ring of R exists and it is simple artinian (cf. [GW04, Corollary 6.19]). Therefore, $\mathcal{Q}_l(R)$ is (isomorphic to) a matrix ring over a division ring and hence it has only one rank function, that we denote by $\dim_{\mathcal{Q}_l(R)}$.

Notice also that the simplicity of R implies that every non-zero left R-module is faithful. In particular, every non-zero finitely generated left R-module is completely faithful. We are going to use Lemma 2.4.1 and transfinite induction to show that for every Sylvester module rank function dim on R and for every finitely generated M with $\mathrm{K.\,dim}(M) < \mathrm{l.\,K.\,dim}(R)$ (equivalently, for every finitely generated torsion module, see [GW04, Proposition 15.7]), we have $\dim(M) = 0$.

The case $\mathrm{K.\,dim}(M) = -1$ follows from (SMod1) and, at every inductive step, if M is finitely generated with $\mathrm{K.\,dim}(M) < \mathrm{l.\,K.\,dim}(R)$, then so is M^k, the direct sum of k copies of M, for every positive integer k (cf. [GW04, Corollary 15.2]). Therefore, Lemma 2.4.1 tells us that M^k contains a cyclic submodule N such that $\mathrm{K.\,dim}(M^k/N) <$

$\mathrm{K.dim}(M^k)$. By induction hypothesis, $\dim(M^k/N) = 0$ and, since N is cyclic, (SMod3) tells us that $\dim(N) \leq 1$. Thus, from the short exact sequence $0 \to N \to M^k \to M^k/N \to 0$, we obtain that $\dim(M) \leq \frac{1}{k}$ for every k in view of (SMod2) and (SMod3). Therefore, the previous claim follows.

Taking into account that, for every non-zero-divisor $x \in R$, right multiplication by x defines an injective endomorphism of left R-modules $R \to R$, we obtain $\mathrm{K.dim}(R/Rx) < \mathrm{l.K.dim}(R)$ by [GW04, Lemma 15.6]. Thus, if rk denotes the Sylvester matrix rank function associated to dim, we deduce

$$\mathrm{rk}(x) = 1 - \dim(R/Rx) = 1$$

Therefore, by Proposition 2.1.9, rk can be extended to $\mathcal{Q}_l(R)$, and by uniqueness of the rank in $\mathcal{Q}_l(R)$, dim must be the rank induced by $\dim_{\mathcal{Q}_l(R)}$. □

We finish the section with another consequence of Lemma 2.4.1 that will prove useful later (see also [Len00, Eisenbud-Robson result, page 131] for the latter part in the simple noetherian case, which can actually be adapted to this situation). We say that a ring R is *almost simple* (as introduced in [Jai99]) if every non-zero two-sided ideal of R contains a non-zero element from its center $Z(R)$. We shall say that a left R-module M is $Z(R)$-*torsionfree* if for all non-zero $c \in Z(R)$ and for all non-zero $m \in M$, we have $cm \neq 0$ (Note that in the language of [GW04, page 81] this would be called a $Z(R)\backslash\{0\}$-torsionfree module).

Proposition 2.4.3. *Let R be a left noetherian almost simple ring with center $Z(R)$, and let* dim *be a Sylvester module rank function on R. If* $\mathrm{l.K.dim}(R) \geq 1$ *and M is a $Z(R)$-torsionfree left R-module of finite length, then* $\dim(M) = 0$.

Proof. Notice first that every non-zero $Z(R)$-torsionfree module M with finite length is completely faithful. Indeed, assume that there exists N M such that M/N is not fully faithful, and let L be a submodule of M with N L and L/N not faithful. Then $\mathrm{ann}_R(L/N)$ is a non-zero two-sided ideal of R, and hence it contains a non-zero element $c \in Z(R)$. Thus, $cL \subseteq N \subsetneq L$. However, since $L \leq M$, we have that L is $Z(R)$-torsionfree of finite length and therefore the map $\varphi_c : L \to L$ defined by left multiplication by c, which is an R-homomorphism because $c \in Z(R)$, is injective. By additivity of length (cf. [GW04, Proposition 4.12]), it must also be surjective, from where $L = \mathrm{im}\,\varphi_c = cL$, a contradiction.

Observe now that for every $k \geq 1$, M^k is also $Z(R)$-torsionfree and has finite length. In particular, $\mathrm{K.dim}(M^k) = 0$ and we can apply Lemma 2.4.1 to deduce that M^k is cyclic. By (SMod3), $\dim(M^k) \leq 1$ for every k, and hence (SMod2) implies that $\dim(M) \leq \frac{1}{k}$ for every k, from where $\dim(M) = 0$. □

2.5 Skew Laurent polynomials over division rings

This section focuses on describing the space of rank functions associated to a skew Laurent polynomial ring $\mathcal{D}[t^{\pm 1}; \tau]$, where $\mathcal{D}$ is a division ring and τ is an automorphism of $\mathcal{D}$.

Unless otherwise specified, throughout this section R denotes a skew Laurent polynomial ring of this form.

The main result of the section describes the extreme rank functions on R and shows that every rank function on R is the unique extension of a rank function on its center $Z(R)$ (cf. Question 1). We also notice at the end of the section that the same theory developed for this ring can be applied to the usual polynomial ring $\mathcal{D}[t]$ and, more generally, to the polynomial rings with coefficients on a simple artinian ring.

In order to prove this, we need to recall the structure of two-sided ideals and the center of R, and we start with the description of the latter in the following lemma (cf. [BK00, Lemma 3.2.14]). Recall that the *inner order* of an automorphism τ is the smallest positive integer m such that τ^m is an inner automorphism, and we say that τ has infinite inner order if no positive power of τ is inner. Over a division ring $\mathcal{D}$, infinite inner order of the automorphim τ of $\mathcal{D}$ is equivalent to the simplicity of $\mathcal{D}[t^{\pm 1};\tau]$ (cf. [GW04, Theorem 1.17]).

Lemma 2.5.1. *Denote $K = Z(\mathcal{D})$, and let K^τ denote the subfield of K formed by the elements of K fixed by the automorphism τ of $\mathcal{D}$. Then,*

(i) If τ has infinite inner order, then $Z(R) = K^\tau$.

(ii) If τ has inner order m, say $\tau^m(d) = a^{-1}da$ for some $a \in \mathcal{D}$, and k is the smallest positive integer for which there exists a non-zero $b \in K$ such that $\tau(ba^k) = ba^k$, then $Z(R) = K^\tau[(ba^k t^{km})^{\pm 1}]$.

Proof. Take an element $p \in Z(R)$, $p = \sum a_i t^i$. Since it must commute with every element $d \in \mathcal{D}$, we obtain that $da_i = a_i\tau^i(d)$ for every i, and since p commutes with t, we have $a_i = \tau(a_i)$. Thus, if τ has infinite inner order, we obtain from the first condition that $a_i = 0$ for every $i \neq 0$ and $a_0 \in K$, and from the second that actually $a_0 \in K^\tau$. Thus, $p \in K^\tau$. Since clearly K^τ lies in the center, we obtain (i).

If, on the contrary, τ has inner order m, then we obtain from the first condition that $a_i = 0$ for every i not dividing m, and that τ^{mj} is given by conjugation by a_{mj} whenever it is non-zero. But τ^{mj} is also given by conjugation by a^j, so for every d,

$$a^{-j}da^j = a_{mj}^{-1}da_{mj}.$$

We deduce that $a_{mj}a^{-j} \in K$, i.e., there exists a non-zero $c_j \in K$ such that $a_{mj} = c_j a^j$. From the second condition we obtain now that $a_0 = c_0 \in K^\tau$, and for every $j \neq 0$, $\tau(c_j a^j) = c_j a^j$, from where the choice of k implies that $k \leq |j|$. We claim that k divides $|j|$ and $c_j b^{-j/k} \in K^\tau$. Indeed, let $\operatorname{sgn}(j)$ denote the sign of j (i.e., $\operatorname{sgn}(j) = 1$ if j is positive and -1 if it is negative), so that $c_j^{\operatorname{sgn}(j)} a^{|j|} = (c_j a^j)^{\operatorname{sgn}(j)}$, and notice that if $|j| = kn + l$ for some $n \geq 1$ and $0 \leq l < k$, then $c_j^{\operatorname{sgn}(j)} b^{-n} \in K$ and

$$\tau(c_j^{\operatorname{sgn}(j)} b^{-n} a^l) = \tau(c_j^{\operatorname{sgn}(j)} a^{|j|}(ba^k)^{-n}) = c_j^{\operatorname{sgn}(j)} a^{|j|}(ba^k)^{-n} = c_j^{\operatorname{sgn}(j)} b^{-n} a^l.$$

The minimality of k implies $l = 0$ and thus $c_j^{\operatorname{sgn}(j)} b^{-n} \in K^\tau$, so $c_j b^{-j/k} \in K^\tau$.

As a consequence, there exists $z_r \in K^\tau$ such that $p = \sum_r z_r b^r a^{kr} t^{mkr}$. Since $(ba^k t^{mk})^r = b^r a^{rk} t^{rmk}$ and, for every $d \in \mathcal{D}$, $d(ba^k t^{mk})^r = (ba^k t^{mk})^r d$, p can be seen as a Laurent polynomial in the commuting variable $ba^k t^{mk}$ with coefficients in K^τ. Conversely, every such polynomial p belongs to the center. Thus, $Z(R)$ is the ordinary Laurent polynomial ring $K^\tau[(ba^k t^{mk})^{\pm 1}]$ and we have proved (ii). □

Note that there always exists an element b as in the hypothesis. Indeed, when τ has finite inner order m we have that R is not simple and, as we would have also deduced from Lemma 2.5.3(1.), $Z(R) \cap \mathcal{D}[t;\tau] \not\subseteq \mathcal{D}$ (For a concrete example, see [GW04, Exercise 1U]). Now, for every $p \in (Z(R) \cap \mathcal{D}[t;\tau]) \setminus \mathcal{D}$, we saw in the proof that every non-zero coefficient corresponding to positive degree mj is of the form $c_j a^j$, $c_j \in K$ and $\tau(c_j a^j) = c_j a^j$, as desired.

More than its precise description, it is important to observe that if R is non-simple, the center of R is an ordinary Laurent polynomial ring over a field, and hence every result in this section also applies to the center. When R is simple, the center is a field and we obtain the following from Proposition 2.4.2.

Corollary 2.5.2. *If τ has infinite inner order, the only Sylvester rank function on R is the one coming from its Ore division ring, and extends the unique Sylvester rank function on $Z(R)$.*

The last assertion of the corollary follows since in $Z(R)$, a field, there exists a unique rank function, which is then necessarily the restriction of the unique rank function in R.

For the rest of the section, assume that τ has finite inner order and let us denote $S = Z(R) \cap \mathcal{D}[t;\tau]$, which is an ordinary polynomial ring over a field by Lemma 2.5.1. In the following lemma we relate two-sided ideals of R and elements of S. Recall that a non-constant $p \in S$ is *irreducible* if it cannot be expressed as a product $p = rq$ for some non-constant $r, q \in S$. Recall also that $p, q \in S$ are said to be *associates* if there exist $r, r' \in S$ such that $p = rq$, $q = r'p$. A comparison of degrees shows that this is the case if and only if $p = dq$ for some unit $d \in Z(R) \cap \mathcal{D}$.

Lemma 2.5.3. *The following hold:*

(1.) *Let I be a non-zero proper two-sided ideal of R. Then, there exists a non-constant polynomial $p \in S$ with non-zero constant term such that p generates I and which is irreducible in S if and only if I is maximal.*

(2.) *There exists a bijective correspondence between maximal two-sided ideals in R and irreducibles in S with non-zero constant term up to association. This defines a bijective correspondence between maximal two-sided ideals of R and maximal ideals of $Z(R)$ sending $\mathfrak{m}$ to $\mathfrak{m}_Z = \mathfrak{m} \cap Z(R)$.*

(3.) *If $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are different maximal two-sided ideals in R, then for all positive integers $k_1, \dots, k_n$, we have $\bigcap_i \mathfrak{m}_i^{k_i} = \mathfrak{m}_1^{k_1} \dots \mathfrak{m}_n^{k_n}$.*

(4.) *Every non-zero proper two-sided ideal I in R is of the form $\bigcap_i \mathfrak{m}_i^{k_i}$ for some maximal two-sided ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ and positive integers $k_1, \dots, k_n$.*

Proof. (1.) Let p be a non-zero element of $I \cap \mathcal{D}[t;\tau]$ of smallest degree, and note that p is non-constant because I is proper. In addition, p has non-zero constant term because otherwise $pt^{-1} \in I \cap \mathcal{D}[t;\tau]$ would be a polynomial of lower degree. Since $\mathcal{D}$ is a division ring, we can take 1 as the constant term. Now, p commutes with t because p and tpt^{-1} have constant term 1 and $(p - tpt^{-1})t^{-1} \in I \cap \mathcal{D}[t;\tau]$ has lower degree than p, and thus it must be zero. Similarly, p commutes with every element of $\mathcal{D}$, and we deduce that $p \in Z(R)$, i.e., $p \in S$.

Since $\mathcal{D}$ is a division ring and τ is an automorphism, we can divide polynomials in $\mathcal{D}[t;\tau]$ (cf. [BK00, 3.2.6]) and hence, given any other $p' \in I$ and an integer k such that $t^k p' \in I \cap \mathcal{D}[t;\tau]$ we have $t^k p' = qp + r$ for some $q, r \in \mathcal{D}[t;\tau]$ with $\deg(r) < \deg(p)$. Since $r \in I \cap \mathcal{D}[t;\tau]$, r must be zero, and thus $p' = (t^{-k}q)p$. Therefore, $I = Rp$ as claimed.

Assume now that I is not maximal and let J be a proper two-sided ideal properly containing I. Then $J = Rq$ for some $q \in S$ as above, and hence $p = rq$ for some $r \in R$. Since p, q have non-zero constant term, a comparison of degrees shows that $r \in \mathcal{D}[t;\tau]$ and $0 < \deg(r), \deg(q) < \deg(p)$. In addition, for every $q' \in R$, we have

$$q'rq = q'p = pq' = rqq' = rq'q$$

from where $q'r = rq'$ since R is a domain. Thus, $r \in S$ and $p = rq$ is a product of non-constant polynomials in S of lower degree and therefore p is not irreducible in S.

Conversely, assume that p is not irreducible in S, and let $r, q \in S$ be non-constant with $p = rq$. Note in particular that $0 < \deg(r), \deg(q) < \deg(p)$, and that q must have non-zero constant term. Therefore $J = Rq$ is a proper two-sided ideal of R with $I \subsetneq J$ (since $\deg(q) < \deg(p)$ and both have non-zero constant term) and hence I is not maximal.

(2.) By (1.), every maximal ideal $\mathfrak{m}$ is generated by an irreducible element $p \in S$ with non-zero constant term. Conversely, if $q \in S$ is irreducible with non-zero constant term we can check as in the final step of (1.) that Rq is a maximal two-sided ideal of R.

Let $\mathfrak{m}_1, \mathfrak{m}_2$ be maximal two-sided ideals in R, and assume $\mathfrak{m}_i = Rp_i$ for irreducible $p_1, p_2 \in S$ with non-zero constant term. If $\mathfrak{m}_1 = \mathfrak{m}_2$, then $p_2 = r_1 p_1$ and $p_1 = r_2 p_2$ for some $r_1, r_2 \in R$. Reasoning as above we can show that $r_i \in S$, and since R is a domain, we obtain from $p_2 = r_1 r_2 p_2$ that they are units of $\mathcal{D}$. Thus p_1 and p_2 are associates in S. Conversely, distinct maximals have non-associate generators, and the correspondence is bijective.

Now, let $Z(R) = K^\tau[s^{\pm 1}]$ with $s = ba^k t^{km}$ as in Lemma 2.5.1. Then $S = K^\tau[s]$ and, since $Z(R)$ is again a Laurent polynomial ring over a field, the same argument above shows that the correspondence between maximal ideals of $Z(R)$ and irreducibles in S with non-zero constant term (up to association) is bijective. Therefore, we have a bijection that sends the maximal two-sided ideal $\mathfrak{m} = Rp$ to the maximal ideal $\mathfrak{m}_Z = Z(R)p$. Moreover, $\mathfrak{m}_Z = \mathfrak{m} \cap Z(R)$, because if $x = rp \in \mathfrak{m} \cap Z(R)$ for some $r \in R$, then the same argument in (1.) shows that $r \in Z(R)$, and hence $x \in \mathfrak{m}_Z$. The other containment is clear.

(3.) Assume that $\mathfrak{m}_i = Rp_i$ for $p_i \in S$ irreducible with non-zero constant term. On the one hand, note that $\mathfrak{m}_1^{k_1} \dots \mathfrak{m}_n^{k_n} = Rp_1^{k_1} \dots p_n^{k_n} \subseteq \cap_i \mathfrak{m}_i^{k_i}$. On the other hand, let

$q \in S$ with non-zero constant term be such that $\cap_i \mathfrak{m}_i^{k_i} = Rq$. In particular, $q = r_1 p_1^{k_1} = \cdots = r_n p_n^{k_n}$ for some $r_1, \ldots, r_n \in R$ and as above we can deduce that $r_i \in S$. But S is an ordinary polynomial ring over a field, hence a unique factorization domain, and $p_1, \ldots, p_n$ are non-associate irreducibles in S. Thus, necessarily $p_1^{k_1} \ldots p_n^{k_n}$ divides q in S, and therefore we also have $\cap_i \mathfrak{m}_i^{k_i} \subseteq \mathfrak{m}_1^{k_1} \ldots \mathfrak{m}_n^{k_n}$.

(4.) If $I = Rp$ for some $p \in S$ with non-zero constant term and $p = up_1^{k_1} \ldots p_n^{k_n}$ in S with $p_i \in S$ non-associate irreducibles and u a unit, then p_i has non-zero constant term and we deduce from the proof of (3.) that $I = \cap_i \mathfrak{m}_i^{k_i}$ where $\mathfrak{m}_i = Rp_i$. □

Before getting to the description of $\mathbb{P}(R)$, we obtain a partial picture by analysing quotient rings $R/\mathfrak{m}^n$ for maximal two-sided ideals $\mathfrak{m}$. Observe that if I is a non-zero proper two-sided ideal generated by a central element $q \in S$ with non-zero constant term, then R/I is a left $\mathcal{D}$-module with $\dim_{\mathcal{D}}(R/I) = \deg(q)$ and with a natural $\mathcal{D}$-basis $\{1 + I, \ldots, t^{\deg(q)-1} + I\}$.

Proposition 2.5.4. *For any maximal two-sided ideal $\mathfrak{m}$ of R and positive integer n, there are exactly n extreme points on $\mathbb{P}(R/\mathfrak{m}^n)$. Moreover, $\mathbb{P}(R/\mathfrak{m}^n) = \mathbb{P}_{\mathrm{reg}}(R/\mathfrak{m}^n)$ and the natural embedding $\varphi_n : Z(R)/(\mathfrak{m}^n \cap Z(R)) \to R/\mathfrak{m}^n$ gives a bijection $\varphi_n^\sharp : \mathbb{P}(R/\mathfrak{m}^n) \to \mathbb{P}(Z(R)/(\mathfrak{m}^n \cap Z(R)))$.*

Proof. By Lemma 2.5.3(1.), there exists $p \in S$ with non-zero constant term and irreducible over S such that $\mathfrak{m} = Rp$. In particular, this implies that $\mathfrak{m}_Z = \mathfrak{m} \cap Z(R) = Z(R)p$ is also a maximal ideal of $Z(R)$ by Lemma 2.5.3(2.).

For $n = 1$, $R/\mathfrak{m}$ is simple because $\mathfrak{m}$ is maximal. It is also left (and hence right; see [GW04, Corollary 4.18]) artinian because every descending chain of left ideals of $R/\mathfrak{m}$ is also a descending chain of left $\mathcal{D}$-modules, and since $R/\mathfrak{m}$ is finite $\mathcal{D}$-dimensional, it must stabilize. Thus, $R/\mathfrak{m}$ is isomorphic to a matrix ring over a division ring, and hence it has only one rank function rk. In this case $Z(R)/\mathfrak{m}_Z$ is a field and hence its unique rank function must then coincide with $\varphi_1^\sharp(\mathrm{rk})$.

For $n \geq 2$, $R/\mathfrak{m}^n$ is a left artinian ring by the previous argument with a unique maximal two-sided ideal $\mathfrak{m}/\mathfrak{m}^n = (p + \mathfrak{m}^n)$. Since $p + \mathfrak{m}^n$ is central nilpotent (with order of nilpotency n), we have $p + \mathfrak{m}^n \subseteq J(R/\mathfrak{m}^n)$ and hence by maximality $J(R/\mathfrak{m}^n) = (p + \mathfrak{m}^n)$. Therefore, $J(R/\mathfrak{m}^n)$ is generated by a central element and $(R/\mathfrak{m}^n)/J(R/\mathfrak{m}^n) = (R/\mathfrak{m}^n)/(\mathfrak{m}/\mathfrak{m}^n) \cong R/\mathfrak{m}$ is simple.

Summing up, $R/\mathfrak{m}^n$ is a left artinian primary ring whose Jacobson radical is generated by a central element of order of nilpotency n. By Corollary 2.2.4, every rank function on $R/\mathfrak{m}^n$ is determined by its values on $p^i + \mathfrak{m}^n$ for $1 \leq i \leq n-1$, and there are exactly n extreme rank functions $\mathrm{rk}_{R/\mathfrak{m}^n,j}$ on $\mathbb{P}(R/\mathfrak{m}^n)$, defined by

$$\mathrm{rk}_{R/\mathfrak{m}^n,j}(p^i + \mathfrak{m}^n) = \begin{cases} \frac{j-i}{j} & \text{if } i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

Now, fix $1 \leq j \leq n$. Any element $q + \mathfrak{m}^n \in R/\mathfrak{m}^n$ gives rise to an endomorphism of the left $\mathcal{D}$-module $R/\mathfrak{m}^j$ given by right multiplication by q. If $\deg(p) = l$, then $R/\mathfrak{m}^j$ has

$\mathcal{D}$-dimension jl as noticed before and hence, fixing any basis of $R/\mathfrak{m}^j$, we have a ring homomorphism $\psi : R/\mathfrak{m}^n \to \mathrm{End}_{\mathcal{D}}(R/\mathfrak{m}^j) \cong \mathrm{Mat}_{jl}(\mathcal{D})$ (recall that the endomorphisms act on the right). Therefore, we can define rank functions $\mathrm{rk}'_{R/\mathfrak{m}^n,j} = \psi^\sharp(\frac{1}{jl}\,\mathrm{rk}_{\mathcal{D}})$ on $R/\mathfrak{m}^n$ and, although the latter isomorphism depends on the choice of the basis, the rank does not, since it is invariant under multiplication by invertible matrices. In particular, the image of $p^i + \mathfrak{m}^n$ with respect to the basis

$$\{1 + \mathfrak{m}^j, \dots, t^{l-1} + \mathfrak{m}^j, p + \mathfrak{m}^j, \dots, t^{l-1}p + \mathfrak{m}^j, \dots, p^{j-1} + \mathfrak{m}^j, \dots, t^{l-1}p^{j-1} + \mathfrak{m}^j\}$$

is the $jl \times jl$ matrix of normalized rank $\frac{j-i}{j}$

$$\begin{pmatrix} 0 & I_{(j-i)l} \\ 0 & 0 \end{pmatrix}$$

when $i \leq j$, and the zero matrix otherwise. Therefore, necessarily $\mathrm{rk}_{R/\mathfrak{m}^n,j} = \mathrm{rk}'_{R/\mathfrak{m}^n,j}$ and the extreme ranks are regular. Since $\mathbb{P}_{\mathrm{reg}}(R/\mathfrak{m}^n)$ is convex, every rank is regular.

Finally, since $Z(R)$ is also a Laurent polynomial ring and $\mathfrak{m}_Z^n = \mathfrak{m}^n \cap Z(R)$, we have the same description of $\mathbb{P}(Z(R)/\mathfrak{m}_Z^n)$, i.e., every rank function is determined by its values on $p^i + \mathfrak{m}_Z^n$ and there are n extreme points $\mathrm{rk}_{Z(R)/\mathfrak{m}_Z^n,j}$ defined as above on these elements. Since φ_n sends $p^i + \mathfrak{m}_Z^n$ to $p^i + \mathfrak{m}^n$, the ranks $\varphi_n^\sharp(\mathrm{rk}_{R/\mathfrak{m}^n,j})$ and $\mathrm{rk}_{Z(R)/\mathfrak{m}_Z^n,j}$ give them the same value, and hence they are equal. Since the extreme points go to the extreme points and $\varphi_n^\sharp$ preserves convex combinations, this finishes the proof. □

As a consequence of the previous classification, we deduce the following result.

Corollary 2.5.5. *Let I be a non-zero proper two-sided ideal of R. If $I = Rp$ with $p \in S$ with non-zero constant term and $p = up_1^{k_1} \dots p_n^{k_n}$ is a factorization of p into non-associate irreducibles of S, then every rank function on R/I is determined by its values on $p_i^j + I$ for $i = 1, \dots, n$, $j = 1, \dots, k_i$.*

Proof. We noticed during the proof of Proposition 2.5.4 that the statement holds when I is a power of a maximal ideal. For the general case, as in the proof of Lemma 2.5.3(4.), $I = \cap_i \mathfrak{m}_i^{k_i}$, where $\mathfrak{m}_i = Rp_i$ are different maximal ideals of R. Since powers of different maximal two-sided ideals are comaximal, we have a ring isomorphism $\varphi : R/I \to \prod_i R/\mathfrak{m}_i^{k_i}$ by the Chinese Remainder Theorem. Take a rank function $\mathrm{rk} \in \mathbb{P}(R/I)$, and assume that $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}')$ for $\mathrm{rk}' \in \mathbb{P}(\prod_i R/\mathfrak{m}_i^{k_i})$.

In view of Proposition 2.1.17, if $\pi_i : \prod_i R/\mathfrak{m}_i^{k_i} \to R/\mathfrak{m}_i^{k_i}$ denotes the i-th projection, rk' can be uniquely written as a convex combination

$$\mathrm{rk}' = \lambda_1 \pi_1^\sharp(\mathrm{rk}_1) + \cdots + \lambda_n \pi_n^\sharp(\mathrm{rk}_n)$$

for some $\mathrm{rk}_i \in \mathbb{P}(R/\mathfrak{m}_i^{k_i})$. Now, by comaximality, $p_i + \mathfrak{m}_s^{k_s}$ is a unit in $R/\mathfrak{m}_s^{k_s}$ for every $s \neq i$, and therefore $\mathrm{rk}_s(p_i^j + \mathfrak{m}_s^{k_s}) = 1$ for every $s \neq i$ and every $j \geq 1$. Thus,

$$\mathrm{rk}(p_i^j + I) = \mathrm{rk}'((p_i^j + \mathfrak{m}_s^{k_s})_s) = \lambda_i \,\mathrm{rk}_i(p_i^j + \mathfrak{m}_i^{k_i}) + (1 - \lambda_i).$$

As a consequence, if we know the values of rk on the previous elements, then on the one hand we can compute the coefficients λ_i via

$$\operatorname{rk}(p_i^{k_i} + I) = \lambda_i \operatorname{rk}_i(p_i^{k_i} + \mathfrak{m}_i^{k_i}) + (1 - \lambda_i) = 1 - \lambda_i$$

and, on the other hand, for every $\lambda_i > 0$, we can compute the values of rk_i on $p_i^j + \mathfrak{m}_i^{k_i}$ for $j = 1, \ldots, k_i - 1$. As in the proof of Proposition 2.5.4, this information determines rk_i uniquely, so we have completely determined rk. □

We have now enough information to determine $\mathbb{P}(R)$ when τ has finite inner order. Let $\mathfrak{m}$ be any maximal two-sided ideal of R. The ring homomorphisms $\pi_{\mathfrak{m},n} : R \to R/\mathfrak{m}^n$ for $n \geq 1$ allow us to define rank functions on R. In particular, if $\mathfrak{m} = Rp$ for some irreducible $p \in S$ with non-zero constant term, then for any positive integer k we have a rank function $\operatorname{rk}_{\mathfrak{m},k}$, which in the notation of Proposition 2.5.4 can be written as $\pi^\sharp_{\mathfrak{m},k}(\operatorname{rk}_{R/\mathfrak{m}^k,k})$, satisfying

$$\operatorname{rk}_{\mathfrak{m},k}(p^i) = \begin{cases} \frac{k-i}{k} & \text{if } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $q \in S$ is irreducible with non-zero constant term and not associated to p, then $\mathfrak{n} = Rq$ defines a different maximal two-sided ideal by Lemma 2.5.3(2.), and hence $\mathfrak{n}+\mathfrak{m}^n = R$ for every $n \geq 1$. Thus, q becomes a unit in every quotient $R/\mathfrak{m}^n$ and therefore $\operatorname{rk}_{\mathfrak{m},k}(q^i) = 1$. In terms of the associated Sylvester module rank functions $\dim_{\mathfrak{m},k}$ we have then for a two-sided maximal ideal $\mathfrak{n}$,

$$\dim_{\mathfrak{m},k}(R/\mathfrak{n}^i) = \begin{cases} \frac{i}{k} & \text{if } \mathfrak{n} = \mathfrak{m} \text{ and } i \leq k \\ 1 & \text{if } \mathfrak{n} = \mathfrak{m} \text{ and } i > k \\ 0 & \text{if } \mathfrak{n} \neq \mathfrak{m} \end{cases}$$

We can also define the rank function rk_0 coming from its Ore quotient ring, whose associated module rank function $\dim_0$ satisfies $\dim_0(R/\mathfrak{n}^i) = 0$ for every maximal two-sided ideal $\mathfrak{n}$ and $i \geq 1$. We are going to prove that, in analogy to the case of Dedekind domains, these are the extreme rank functions on R. For this, we note first the following analog of Lemma 2.3.3, which is proved similarly invoking Lemma 2.5.3(3.).

Lemma 2.5.6. *Let* dim *be a Sylvester module rank function on* $R = \mathcal{D}[t^{\pm 1}; \tau]$. *There exist only countably many maximal two-sided ideals* $\mathfrak{m}$ *of* R *such that* $\dim(R/\mathfrak{m}^k) > 0$ *for some* $k \geq 1$.

Secondly, from Lemma 2.5.3(1.) we know that R is almost simple, and we observed in Section 2.4 that $\operatorname{l.K.dim}(R) = 1$. Hence, we can use Proposition 2.4.3 to deduce in the next proposition that the rank of a finite $\mathcal{D}$-dimensional R-module M is determined by its $Z(R)$-torsion submodule

$$t_{Z(R)}(M) = \{m \in M : cm = 0 \text{ for some non-zero } c \in Z(R)\}.$$

Note that in the language of [GW04, page 83] this would be called $Z(R)\setminus\{0\}$-torsion submodule, and it is actually a submodule because R is a domain and $Z(R)$ is commutative.

Proposition 2.5.7. *If M is a finitely generated R-module with $\dim_{\mathcal{D}}(M) < \infty$, then, for any Sylvester module rank function $\dim \in \mathbb{P}(R)$, we have $\dim(M) = \dim(t_{Z(R)}(M))$.*

Proof. Denote $N = t_{Z(R)}(M)$ and let dim be a Sylvester module rank function on R. If $N = 0$, then M is $Z(R)$-torsionfree of finite length (since it is finite $\mathcal{D}$-dimensional) and hence $\dim(M) = 0 = \dim(N)$ by Proposition 2.4.3. Assume now that N is non-zero. Since R is noetherian, N is also finitely generated, say, by $\{m_1, \dots, m_n\}$. For every k, we can take a non-zero $p_{m_k} \in Z(R) \cap \mathcal{D}[t;\tau]$ with non-zero constant term that annihilates m_k and observe that, if $p = \prod_k p_{m_k}$, then $N = \{m \in M : pm = 0\}$. Since $p \in Z(R)$, pM is a submodule of M, and we claim that $M = N \oplus pM$.

Indeed, their intersection $N \cap pM$ is trivial, because if $pm \in N$, then $p^2 m = 0$, from where $m \in N$ (by definition, since $0 \neq p^2 \in Z(R)$), and therefore $pm = 0$. Now, from the exact sequence (of $\mathcal{D}$-modules) $0 \to N \to M \to pM \to 0$, where the homomorphism $M \to pM$ is given by left multiplication by p,, we obtain that $\dim_{\mathcal{D}}(M) = \dim_{\mathcal{D}}(N) + \dim_{\mathcal{D}}(pM)$, and hence, since from the second isomorphism theorem and additivity of $\dim_{\mathcal{D}}$,

$$\dim_{\mathcal{D}}(N + pM) + \dim_{\mathcal{D}}(N \cap pM) = \dim_{\mathcal{D}}(N) + \dim_{\mathcal{D}}(pM) = \dim_{\mathcal{D}}(M),$$

we deduce that $M = N \oplus pM$. Since pM is $Z(R)$-torsionfree of finite length, Proposition 2.4.3 gives that $\dim(pM) = 0$, and therefore $\dim(M) = \dim(N)$ by (SMod2). $\square$

Theorem 2.5.8. *The extreme points on $\mathbb{P}(R)$ are precisely the rank functions $\dim_{\mathfrak{m},k}$ and $\dim_0$ defined above, and any other rank function can be uniquely expressed as a (possibly infinite) convex combination of them. Thus, $\mathbb{P}(R) = \mathbb{P}_{\mathrm{reg}}(R)$. Moreover, the inclusion map $\iota : Z(R) \to R$ gives a bijection $\iota^\sharp : \mathbb{P}(R) \to \mathbb{P}(Z(R))$.*

Proof. Consider a Sylvester module rank function dim on R. Let S_0 be the set of two-sided maximal ideals $\mathfrak{m}$ for which there exists k with $\dim(R/\mathfrak{m}^k) > 0$, which is countable by Lemma 2.5.6. We are going to show that there exist non-negative real numbers $c_0, c_{\mathfrak{m},k}$ satisfying

$$\dim = c_0 \dim_0 + \sum_{\mathfrak{m}\in S_0}\left(\sum_{k\in\mathbb{Z}^+} c_{\mathfrak{m},k} \dim_{\mathfrak{m},k}\right) \quad \text{and} \quad c_0 + \sum_{\mathfrak{m}\in S_0}\left(\sum_{k\in\mathbb{Z}^+} c_{\mathfrak{m},k}\right) = 1.$$

Set $b_{\mathfrak{m},0} = \dim(R/\mathfrak{m})$ and $b_{\mathfrak{m},k} = \dim(R/\mathfrak{m}^{k+1}) - \dim(R/\mathfrak{m}^k)$ for every $k \geq 1$. Since $\mathfrak{m} = Rp$ for some irreducible $p \in S = Z(R) \cap \mathcal{D}[t;\tau]$ and $\mathfrak{m}^k = Rp^k$, Lemma 1.2.4 implies that $b_{\mathfrak{m},k} \geq b_{\mathfrak{m},k+1}$ for every $k \geq 0$. Thus, if we impose equality of both expressions on the modules $R/\mathfrak{n}^i$ for maximal two-sided ideals $\mathfrak{n}$ and $i \geq 1$, we can reason as in Theorem 2.3.5 to show that the only possible solution is given by the non-negative coefficients

$$c_{\mathfrak{m},k} = k(b_{\mathfrak{m},k-1} - b_{\mathfrak{m},k}), k \geq 1$$

and that $c_0 = 1 - \sum_{\mathfrak{m}\in S_0}\sum_{k\in\mathbb{Z}^+} c_{\mathfrak{m},k}$ is well-defined. Nevertheless, we do not know in principle that a rank function on R is already determined by its values on those modules, so we still need to check that this is sufficient to claim equality. Define

$$\dim' := c_0 \dim_0 + \sum_{\mathfrak{m}\in S_0}\sum_{k\in\mathbb{Z}^+} c_{\mathfrak{m},k}\dim_{\mathfrak{m},k}$$

We have just seen that $\dim'$ is a Sylvester module rank function on R that coincides with dim on the modules $R/\mathfrak{n}^k$ for every maximal two-sided ideal $\mathfrak{n}$ and $k \geq 1$.

Take now any non-zero proper two-sided ideal I such that $d_I = \dim(R/I) > 0$, and notice that $\dim'(R/I) = d_I$ (by using Lemma 2.5.3(4.), the Chinese Remainder Theorem and that dim and $\dim'$ coincide on quotients by powers of maximal ideals). Now, $\frac{1}{d_I}\dim$ and $\frac{1}{d_I}\dim'$ define Sylvester module rank functions on R/I. Indeed, any finitely presented R/I-module is a finitely generated R-module with the natural operation, and hence finitely presented because R is noetherian. Thus, they satisfy (SMod1)-(SMod3) because dim and $\dim'$ do. But if $I = Rq$ for $q \in S$ with non-zero constant term and $q = uq_1^{k_1}\dots q_n^{k_n}$ for some non-associate irreducibles $q_i \in S$ and a unit $u \in S$, then rank functions on R/I are determined by their value on $q_i^j + I$ for $i = 1,\dots,n$, $j = 1,\dots,k_i$ by Corollary 2.5.5. Equivalently, if $\mathfrak{n}_i = Rq_i$, they are determined by their value on the modules $(R/I)/(R/I(q_i^j + I)) = (R/I)/(\mathfrak{n}_i^j/I) \cong R/\mathfrak{n}_i^j$ for $1 \leq i \leq n$, $1 \leq j \leq k_i$. Therefore, by construction, $\frac{1}{d_I}\dim = \frac{1}{d_I}\dim'$ as rank functions on R/I.

Let M be a finitely generated left R-module with $\dim_{\mathcal{D}}(M) < \infty$. By Proposition 2.5.7, if $N = t_{Z(R)}(M)$, then $\dim(M) = \dim(N)$ and $\dim'(M) = \dim'(N)$. Thus, if $N = 0$ then $\dim(M) = 0 = \dim'(M)$. If N is non-zero, then as in the proof of Proposition 2.5.7 there exists $q \in S$ with non-zero constant term such that $N = \{m \in M : qm = 0\}$ and $M = N \oplus qM$. Define $I = Rq$, and note that N is a finitely presented R/I-module. If $d_I = 0$ and N is a k-generated R-module, we obtain from the surjective R-homomorphism $(R/I)^k \to N$, (SMod2) and (SMod3) that $\dim(N) = \dim'(N) = 0$. Otherwise, the preceding paragraph shows that $\dim(N) = d_I(\frac{1}{d_I}\dim(N)) = d_I(\frac{1}{d_I}\dim'(N)) = \dim'(N)$. Therefore, $\dim(M) = \dim'(M)$.

Finally, take any matrix A over R. Since adding rows and columns of zeros do not change the rank, we can assume that A is $n \times n$. Take k such that $A(I_n t^k)$ is a matrix over $\mathcal{D}[t;\tau]$. Thus, there exist $n \times n$ invertible matrices P, Q over $\mathcal{D}[t;\tau]$ respectively, and a diagonal matrix D, such that $A = PDQ(I_n t^{-k})$. (cf. [BK00, Proposition 3.2.8 & 3.3.2]). Thus, any Sylvester matrix rank function on R is determined by its values on elements, or equivalently any Sylvester module rank function is determined by its values on the quotients R/Rp for $p \in R$, which are either free if $p = 0$ or have finite dimension over $\mathcal{D}$. Since we have seen that dim and $\dim'$ coincide on these modules, $\dim = \dim'$.

Since $Z(R)$ is a Laurent polynomial ring over a field, we have the same classification of rank functions on $Z(R)$. In view of Lemma 2.5.3(2.), for every maximal two-sided ideal $\mathfrak{m}$ and every positive integer k, we have an extreme point $\mathrm{rk}_{\mathfrak{m}_Z,k}$ in $\mathbb{P}(Z(R))$ where $\mathfrak{m}_Z = \mathfrak{m} \cap Z(R)$. Moreover, if we denote by $\pi_{\mathfrak{m}_Z,k} : Z(R) \to Z(R)/\mathfrak{m}_Z^k$ the canonical homomorphism, then from the definition of the extreme rank functions and using

Proposition 2.5.4, we have

$$\mathrm{rk}_{\mathfrak{m}_Z,k} = \pi^\sharp_{\mathfrak{m}_Z,k}(\mathrm{rk}_{Z(R)/\mathfrak{m}_Z^k,k}) = \pi^\sharp_{\mathfrak{m}_Z,k}\varphi^\sharp_k(\mathrm{rk}_{R/\mathfrak{m}^k,k}) = \iota^\sharp\pi^\sharp_{\mathfrak{m},k}(\mathrm{rk}_{R/\mathfrak{m}^k,k}) = \iota^\sharp(\mathrm{rk}_{\mathfrak{m},k})$$

The other extreme point in $\mathbb{P}(Z(R))$ is $\mathrm{rk}_{Z,0}$, the one induced by the unique rank function on its field of fractions $\mathcal{Q}(Z(R))$. By uniqueness, the rank function on $\mathcal{Q}(Z(R))$ is the restriction of the unique rank function on $\mathcal{Q}_l(R)$, and hence $\mathrm{rk}_{Z,0} = \iota^\sharp(\mathrm{rk}_0)$. Thus, the map $\iota^\sharp$ sends the extreme points to the extreme points, and since it preserves (infinite) convex combinations it is bijective. This finishes the proof. □

Notice that the same analysis done for $\mathcal{D}[t^{\pm 1};\tau]$ works for ordinary polynomials $\mathcal{D}[t]$ over a division ring $\mathcal{D}$, since $\mathcal{D}[t]$ enjoys the same properties that we have used for skew Laurent polynomials. Namely, $\mathcal{D}[t]$ is a noetherian domain with $\mathrm{l.K.dim}(\mathcal{D}[t]) = 1$ in which every two-sided ideal is generated by an element in $Z(\mathcal{D}[t]) = Z(\mathcal{D})[t]$ (cf. [GW04, Theorem 1.9, Theorem 15.17 & Proposition 17.1(c)]). From the latter property we can reason as in Lemma 2.5.3 to directly see that, if $K = Z(\mathcal{D})$, there is a bijection between maximal two-sided ideals in $\mathcal{D}[t]$ and maximal ideals in $K[t]$, sending $\mathfrak{m} = Rp$ for some non-zero $p \in K[t]$ to $\mathfrak{m}_Z = \mathfrak{m} \cap K[t] = K[t]p$, and we can also deduce that every non-zero proper two-sided ideal can be written as intersection (equivalently, product) of powers of maximal ideals. With this, the proofs of Proposition 2.5.4, Corollary 2.5.5, Lemma 2.5.6 Proposition 2.5.7 and Theorem 2.5.8 apply with the corresponding changes to this case. Thus, we have the following partial answer to Question 1 for simple artinian rings.

Proposition 2.5.9. *Let R be a simple artinian ring. The inclusion $\iota : Z(R)[t] \to R[t]$ defines a bijection $\iota^\sharp : \mathbb{P}(R[t]) \to \mathbb{P}(Z(R)[t])$. In particular, every Sylvester rank function on $Z(R)[t]$ can be uniquely extended to a Sylvester rank function on $R[t]$.*

Proof. Since R is simple artinian, there exists a division ring $\mathcal{D}$ and a positive integer n such that $R \cong \mathrm{Mat}_n(\mathcal{D})$. The inclusion map ι factors through

$$Z(R)[t] \to Z(\mathcal{D})[t] \hookrightarrow \mathcal{D}[t] \to \mathrm{Mat}_n(\mathcal{D}[t]) \to R[t],$$

where the first map is the extension of the isomorphism $Z(R) \cong Z(\mathcal{D})$, the third one is the diagonal embedding and the last one is induced by $\mathrm{Mat}_n(\mathcal{D}[t]) \cong \mathrm{Mat}_n(\mathcal{D})[t]$. Consequently, $\iota^\sharp$ factors as

$$\mathbb{P}(R[t]) \to \mathbb{P}(\mathrm{Mat}_n(\mathcal{D}[t])) \to \mathbb{P}(\mathcal{D}[t]) \to \mathbb{P}(Z(\mathcal{D})[t]) \to \mathbb{P}(Z(R)[t]).$$

The first and the last map are induced by ring isomorphisms and hence bijective. The second one is bijective by Proposition 2.1.14, and the third one is bijective by the analog of Theorem 2.5.8. □

As a final remark, note that we cannot expect the same result to hold for skew polynomial rings $\mathcal{D}[t;\tau]$. For example, if τ has infinite inner order, then $Z(\mathcal{D}[t;\tau]) = K^\tau$ for $K = Z(\mathcal{D})$, and thus $\mathbb{P}(Z(\mathcal{D}[t;\tau])) = \{\dim_{K^\tau}\}$, while the natural maps $\mathcal{D}[t;\tau] \to \mathcal{Q}(\mathcal{D}[t;\tau])$ and $\mathcal{D}[t;\tau] \to \mathcal{D}$ define two different Sylvester rank functions on the skew polynomial ring.

Chapter 3

Epic division rings and pseudo-Sylvester domains

In this chapter, which is based on [HL20], we first introduce universal localizations and epic division rings in Section 3.1, together with the main results in P.M. Cohn's classification of homomorphisms from a given ring R to division rings.

There are two instances of epic division rings that are of particular interest in this and the upcoming chapters, namely, the universal division ring and the Hughes-free division ring of fractions. Given a ring R, we say that an epic division R-ring $\mathcal{D}$ is universal, another concept introduced by P.M. Cohn, if every matrix over R that becomes invertible under a homomorphism to a division ring is also invertible in $\mathcal{D}$. The Hughes-free division ring of fractions was introduced by I. Hughes in [Hug70] and it is defined in the context of crossed products $R = E * G$ of a division ring E and a locally indicable group G. The existence of a universal or a Hughes-free division ring of fractions is not guaranteed in general, but if they exist, they are unique up to R-isomorphism.

With respect to universal division rings of fractions, we recover and study here again Sylvester and pseudo-Sylvester domains in Section 3.2. Although their definition in Definition 1.1.9 may seem to be unrelated to division rings, they are the families of rings that admit a universal division ring of fractions in which every matrix that one can naively expect to become invertible actually becomes invertible. We make this statement precise during this chapter and develop a homological criterion to recognize pseudo-Sylvester domains based on that of A. Jaikin-Zapirain for Sylvester domains ([Jai20C]). We then study the case of a crossed product $\mathfrak{F} * \mathbb{Z}$ of a fir $\mathfrak{F}$ and the ring of integers $\mathbb{Z}$ in Section 3.5. This part was a joint work with Fabian Henneke in [HL20].

With respect to Hughes-free division rings of fractions, we introduce them in Section 3.4 and show that the Hughes-free property can be restated in terms of Sylvester matrix rank functions. We also sketch the relation between the Hughes-free and the universal division ring of fractions, and use it later to give a particular realization of the universal division $E * G$-ring of fractions, where E is a division ring and G is a group with a normal free subgroup F such that G/F is infinite cyclic. This notion will show up again in Chapter 4 in the context of the strong Atiyah conjecture, and there we settle the existence of the Hughes-free division ring of fractions for the case of group rings $K[G]$ of a locally indicable group G over a field of characteristic zero. We also explore further in Chapter 5 the relation of Hughes-free and universal division rings of fractions.

3.1 Universal localization and epic division rings. Universal division rings of fractions

We start this section by recalling the construction of the localization of a ring with respect to a left Ore set. We follow the exposition in [GW04].

Recall that whenever we have a commutative ring R and a prime ideal $\mathfrak{p}$ of R, one can consider the localization of R at $\mathfrak{p}$, usually denoted $R_{\mathfrak{p}}$, a local ring in which every element outside $\mathfrak{p}$ becomes invertible. Moreover, every element of $R_{\mathfrak{p}}$ can be expressed as a fraction of the form $t^{-1}r$ for some $r \in R$ and $t \in R \backslash \mathfrak{p}$. When R is a domain and $\mathfrak{p}$ is the zero ideal, this leads to the construction of its field of fractions.

When R is non-commutative, one can ask whether there exists a ring with such a description in which the denominators are taken from a certain subset $T \subseteq R$ of non-zero-divisors. We can always assume that T is multiplicatively closed and contains the unity, since products of elements of T would also become invertible in the new ring. In this case, we say that T is a *multiplicative set*.

Definition 3.1.1. Let R be a ring and let $T \subseteq R$ be a multiplicative set of non-zero-divisors. A *left ring of fractions* for R with respect to T is a ring S in which R embeds and that satisfies the following:

1. Every element of T becomes invertible in S.

2. Every element of S can be written as $t^{-1}r$ for some $t \in T$ and $r \in R$.

Right rings of fractions are defined symmetrically.

Observe that a left ring of fractions may not exist in general, since we have no way in principle to ensure that sums and products of elements of the form $t^{-1}r$ admit a similar expression. The condition that guarantee the feasibility of this procedure is the so-called Ore condition.

Definition 3.1.2. Let R be a ring and let $T \subset R$ be a multiplicative set. We say that T *satisfies the left Ore condition* or that T is a *left Ore set* if, for every $t \in T$ and $r \in R$, $Tr \cap Rt \neq \varnothing$. Similarly, we define *right Ore sets*, and we say that T is an *Ore set* if it satisfies both the left and the right Ore conditions.

If we want to construct a ring whose elements are as above, the Ore condition allows us to express a wrong-side fraction rt^{-1} in the appropriate way: if $t' \in T$ and $r' \in R$ are such that $t'r = r't$ in R, then in this hypothetical ring we would have $rt^{-1} = (t')^{-1}r'$. But in fact, if the set T contains only non-zero-divisors, this condition is also sufficient. We record here the result and the construction of the left ring of fractions (see the discussion in [GW04] after Lemma 6.1, and Theorem 6.2).

Theorem 3.1.3. *Let R be a ring and let $T \subset R$ be a multiplicative set of non-zero-divisors. Then there exists a left ring of fractions with respect to T if and only if T is a left Ore set. In this case, a left ring of fractions, which we denote by $T^{-1}R$, can be constructed as follows:*

1. *Define the equivalence relation in $T \times R$ given by $(t, r) \sim (t', r')$ if and only if there exist $s, s' \in R$ such that $sr = s'r'$ and $st = s't' \in T$. Denote the equivalence class of (t, r) by $t^{-1}r$ and let $T^{-1}R$ be the set of these equivalence classes.*

2. *Since T is a multiplicative left Ore set, given two classes $t_1^{-1}r_1$ and $t_2^{-1}r_2$ there exist $s_1, s_2 \in R$ such that $s_1t_1 = s_2t_2 \in T$. Set*
$$t_1^{-1}r_1 + t_2^{-1}r_2 = (s_1t_1)^{-1}(s_1r_1 + s_2r_2).$$

3. *Since T is a left Ore set, given two classes $t_1^{-1}r_1$ and $t_2^{-1}r_2$, there exist $t \in T$, $r \in R$ such that $tr_1 = rt_2$. Set*
$$t_1^{-1}r_1 \cdot t_2^{-1}r_2 = (tt_1)^{-1}rr_2.$$

These operations are well defined and $T^{-1}R$ becomes a ring in which R embeds through the ring homomorphism $R \to T^{-1}R$ given by $r \mapsto 1^{-1}r$.

As in the commutative case, a left ring of fractions satisfies a universal property. From this property it is deduced in particular that a left ring of fractions is unique up to isomorphism and that if T is also right Ore, then $T^{-1}R = RT^{-1}$ (cf. [GW04, Proposition 6.3 & Proposition 6.5]).

Proposition 3.1.4. *Let R be a ring, $T \subseteq R$ a left Ore set of non-zero-divisors, and S a left ring of fractions of R with respect to T. If $\varphi : R \to R'$ is a ring homomorphism and $\varphi(t)$ is invertible for every $t \in T$, then φ extends uniquely to a ring homomorphism $\tilde{\varphi} : S \to R'$.*
In particular, there exists an isomorphism $S \cong T^{-1}R$, and if T is also a right Ore set, then $T^{-1}R = RT^{-1}$.

Observe that if R is a domain and the set of all non-zero elements of R is a left Ore set, then the left ring of fractions of R with respect to that set is a division ring in which R embeds. Let us introduce some extra definitions and notation in regard to this and the previous proposition.

Definition 3.1.5. Let R be a ring.

a) If $T \subseteq R$ is a (multiplicative) left Ore set of non-zero-divisors, then we call $T^{-1}R$ the *left Ore localization* of R with respect to T. If T is also right Ore, we call $T^{-1}R$ the *Ore localization* of R with respect to T.

b) If the set T of all non-zero-divisors in R is left Ore, then $T^{-1}R$ is called the *classical left quotient ring of R* and it is denoted by $\mathcal{Q}_l(R)$. If T is also right Ore, then we call $T^{-1}R$ the *classical quotient ring of R* and we denote it by $\mathcal{Q}(R)$.

c) If R is a domain and the set of all non-zero elements in R is left Ore, then we say that R is a *left Ore domain.* In this case, $\mathcal{Q}_l(R)$ is a division ring and it is usually called the *left Ore quotient ring* or *left Ore division ring.* *Right Ore domains* are defined similarly, and if R is both a left and a right Ore domain we say that R is an *Ore domain* and call $\mathcal{Q}(R)$ its *Ore division ring.*

Remark 3.1.6. Recall from Chapter 1 that the embedding $R \to T^{-1}R$ to some left Ore localization of R is epic. Indeed, if we have a ring homomorphism $\phi : T^{-1}R \to Q$ to some ring Q, then necessarily $\phi(t^{-1}r) = \phi(t)^{-1}\phi(r)$, i.e., ϕ is completely determined by its values on R. □

The following example was already mentioned during Chapter 1.

Example 3.1.7. Let R be a ring, and let τ be an automorphism of R. Then,

- The set of powers of t in the skew polynomial ring $R[t;\tau]$ is a multiplicative Ore set. Indeed, for every $p = \sum a_i t^i \in R[t;\tau]$ and every non-negative integer k, there exist the polynomials $q = \sum \tau^k(a_i)t^i$ and $q' = \sum \tau^{-k}(a_i)t^i$ such that $t^k p = qt^k$ and $pt^k = t^k q'$.

- The Ore localization of $R[t;\tau]$ with respect to the powers of t is the skew Laurent polynomial ring $R[t^{\pm 1};\tau]$. Indeed, $R[t^{\pm 1};\tau]$ is an overring of $R[t;\tau]$ in which the powers of t are invertible and every element can be written as $t^{-k}p$ for some non-negative integer k and $p \in R[t;\tau]$.

- If R is a left (resp. right) noetherian domain, then both $R[t;\tau]$ and $R[t^{\pm 1};\tau]$ are left (resp. right) Ore domains (cf. [GW04, Theorem 1.14, Corollary 1.15 & Corollary 6.7]). In particular, if $R = \mathcal{D}$ is a division ring, then $\mathcal{D}[t;\tau]$ is an Ore domain and its Ore division ring is sometimes denoted $\mathcal{D}(t;\tau)$.

□

We finish the introduction to left Ore localizations $T^{-1}R$ by stating its most important homological property: it is a flat right R-module (cf. [GW04, Corollary 10.13]), so the functor $T^{-1}R \otimes_R \square$ is an exact functor.

Proposition 3.1.8. *Let R be a ring and let $T \subseteq R$ be a (multiplicative) left Ore set of non-zero-divisors. Then $T^{-1}R$ is flat as a right R-module.*

We have seen so far that for a non-commutative ring R to be embedded in a division ring whose description resembles that of the field of fractions in the commutative setting, R must be a left or right Ore domain. For the rest of the section, we are going to consider the more general question of whether a given non-commutative domain R can be embedded at all into a division ring.

In this full generality, it can be treated by means of P. M. Cohn's theory of epic division R-rings (cf. [Coh06, Chapter 7]), which relies on the existence of prime matrix ideals (for the definition of this notion, we refer the reader to [Coh06, Chapter 7, Section 3]), and universal localizations. This latter notion generalizes Ore localization in the sense that we do not choose a multiplicative set of elements to be inverted but a set of square matrices, as follows.

Definition 3.1.9. Given a set Σ of (square) matrices over R, and a homomorphism of rings $\varphi : R \to S$, we say that the map φ is Σ*-inverting* if every element of Σ becomes invertible over S. We say that φ is *universal* Σ*-inverting* if any other Σ-inverting homomorphism factors uniquely through φ. In this latter case, we denote $S = R_\Sigma$ and we call R_Σ *the universal localization of* R *with respect to* Σ.

If we allow R_Σ to be the zero ring, the existence of the universal localization can always be proved by taking a presentation of R as a ring and formally adding the necessary generators and relations (see the discussion in [Coh06] before Theorem 7.2.4, or consult [Sán08, Theorem 3.23]). Moreover, in principle we can allow Σ to contain non-square matrices, but recall that over IBN rings only square matrices can be invertible and hence, since we are interested in homomorphisms to division rings, we shall restrict, after this point, to square matrices. We have the following (cf. [Coh06, Theorem 7.2.4]).

Proposition 3.1.10. *Let R be a ring and let Σ be a set of matrices over R. Then the universal localization R_Σ exists, it is unique up to isomorphism, and if $\varphi : R \to R_\Sigma$ is the universal Σ-inverting homomorphism, then φ is injective if and only if there exists a Σ-inverting embedding of R into some ring.*

Remark 3.1.11. As mentioned above, the universal localization can be abstractly constructed from a presentation of R by adjoining, for every $n \times m$ matrix $A = (a_{ij}) \in \Sigma$, nm generators b_{ij} with defining relations $BA = I_m$ and $AB = I_n$, where $B = (b_{ij})$. The universal Σ-inverting map $\varphi : R \to R_\Sigma$ just send the generators of R to their copy in R_Σ.

From here, observe that φ is epic. Indeed, assume that we have homomorphisms $f, g : R_\Sigma \to S$ with $f \circ \varphi = g \circ \varphi$. In particular, if $A \in \Sigma$, then $f(\varphi(A)) = g(\varphi(A))$, and therefore, if B denotes the inverse of $\varphi(A)$ in R_Σ,

$$f(B) = f(\varphi(A))^{-1} = g(\varphi(A))^{-1} = g(B).$$

This means that f and g also coincide on the other generators (b_{ij}) added to define R_Σ, and therefore, $f = g$. □

To explain P.M. Cohn's main result on the topic, let us introduce R-rings and epic division R-rings.

Definition 3.1.12. Let R be a ring.

i) An *R-ring* is a pair (S, φ) where S is a ring and $\varphi : R \to S$ is a ring homomorphism. We say that two R-rings (S_1, φ_1) and (S_2, φ_2) are *R-isomorphic* (or simply *isomorphic*) if there exists a ring isomorphism $\psi : S_1 \to S_2$ such that the following diagram commutes.

$$\begin{tikzcd} & R \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ S_1 \arrow[rr, "\psi"] & & S_2 \end{tikzcd}$$

ii) An *epic division R-ring* (or *epic R-field*) is an R-ring $(\mathcal{D}, \varphi)$ where $\mathcal{D}$ is a division ring generated, as a division ring, by the image of R. It is further called *division*

R-ring of fractions if φ is injective, and in this case we shall sometimes omit the map and just say that $R \hookrightarrow \mathcal{D}$ is a division R-ring of fractions.

Although the definition given here is more practical, the epic terminology is not just a coincidence ([Coh06, Corollary 7.2.2]).

Proposition 3.1.13. *Let R be a ring and $\varphi : R \to \mathcal{D}$ a homomorphism from R to a division ring $\mathcal{D}$. Then φ is epic if and only if $\mathcal{D}$ is generated, as a division ring, by $\varphi(R)$.*

P.M. Cohn proved that epic division R-rings are completely characterized (up to R-isomorphism) by the set Σ of matrices over R that become invertible in the division ring, and that they always arise as residue-class division rings of the universal localizations R_Σ. More precisely, if we combine (cf. [Coh06, Theorem 7.2.5(ii) & Theorem 7.2.7], we obtain the following.

Theorem 3.1.14. *Let R be a ring, $(\mathcal{D}, \varphi)$, $(\mathcal{D}', \varphi')$ two epic division R-rings and let Σ, Σ' be the sets of square matrices over R that become invertible over $\mathcal{D}$ and $\mathcal{D}'$, respectively. The following hold:*

a) The universal localization R_Σ is a local ring whose residue-class division ring is isomorphic to $\mathcal{D}$.

b) $(\mathcal{D}, \varphi)$ and $(\mathcal{D}', \varphi')$ are R-isomorphic if and only if $\Sigma = \Sigma'$.

Observe that we can write Theorem 3.1.14 *b*) in terms of Sylvester matrix rank functions as follows.

Corollary 3.1.15. *Let R be a ring, $(\mathcal{D}, \varphi)$, $(\mathcal{D}', \varphi')$ two epic division R-rings, and let $\mathrm{rk}_\mathcal{D}$, $\mathrm{rk}_{\mathcal{D}'}$ be the usual ranks in $\mathcal{D}$ and $\mathcal{D}'$, respectively. Then $(\mathcal{D}, \varphi)$ and $(\mathcal{D}', \varphi')$ are R-isomorphic if and only if $\varphi^\sharp(\mathrm{rk}_\mathcal{D}) = (\varphi')^\sharp(\mathrm{rk}_{\mathcal{D}'})$, i.e., if for every matrix A over R, we have*

$$\mathrm{rk}_\mathcal{D}(\varphi(A)) = \mathrm{rk}_{\mathcal{D}'}(\varphi'(A)).$$

Proof. Assume first that $(\mathcal{D}, \varphi)$ and $(\mathcal{D}', \varphi')$ are R-isomorphic and let $\psi : \mathcal{D} \to \mathcal{D}'$ denote an R-isomorphism, so that $\psi \circ \varphi = \varphi'$. By uniqueness of the rank function on $\mathcal{D}$, we must have $\psi^\sharp(\mathrm{rk}_{\mathcal{D}'}) = \mathrm{rk}_\mathcal{D}$, and therefore

$$\varphi^\sharp(\mathrm{rk}_\mathcal{D}) = \varphi^\sharp(\psi^\sharp(\mathrm{rk}_{\mathcal{D}'})) = (\varphi')^\sharp(\mathrm{rk}_{\mathcal{D}'}).$$

Conversely, assume that $\varphi^\sharp(\mathrm{rk}_\mathcal{D}) = (\varphi')^\sharp(\mathrm{rk}_{\mathcal{D}'})$ and let Σ and Σ' denote the sets of square matrices over R that become invertible over $\mathcal{D}$ and $\mathcal{D}'$, respectively. If $A \in \Sigma$ is an $n \times n$ matrix, then $\varphi(A)$ is invertible and therefore $\mathrm{rk}_\mathcal{D}(\varphi(A)) = n$. Thus, $\mathrm{rk}_{\mathcal{D}'}(\varphi'(A)) = n$ and $\varphi'(A)$ is invertible. Thus, $A \in \Sigma'$ and we have proved that $\Sigma \subseteq \Sigma'$. Similarly, we get the other containment and hence the equality $\Sigma = \Sigma'$. From Theorem 3.1.14 *b*), the epic division rings are R-isomorphic. $\square$

P.M. Cohn actually proved more than this. Clearly not every set Σ of square matrices over R is susceptible to appear as the set of matrices that become invertible under a ring homomorphism $R \to \mathcal{D}$. He identified such sets precisely as the complements, in the set of square matrices over R, of prime matrix ideals $\mathcal{P}$ ([Coh06, Theorem 7.4.3 & Theorem 7.2.5(i)]). We shall not introduce prime matrix ideals, but we rather remark that Malcolmson proved in [Mal80] that there exists a bijective correspondence between prime matrix ideals on R and integer-valued Sylvester matrix rank functions rk on R. The actual statement (just in term of maps of sets) of [Mal80, Theorem 2] is the following.

Theorem 3.1.16. *Let R be a ring. The correspondence that associates to each epic division R-ring $(\mathcal{D}, \varphi)$ the Sylvester matrix rank function $\varphi^\sharp(\mathrm{rk}_{\mathcal{D}})$ on R gives a bijection between the set of epic division R-rings (up to R-isomorphism) and the set of integer-valued Sylvester matrix rank functions on R.*

The next corollary is just a rewriting of the "surjective" part from the previous theorem, which motivates the subsequent definitions.

Corollary 3.1.17. *Let R be a ring. If* rk *is an integer-valued Sylvester matrix rank function on R, then there exists a unique (up to R-isomorphism) epic division R-ring $(\mathcal{D}, \varphi)$ such that* $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}_{\mathcal{D}})$.

Definition 3.1.18. Let R be a ring. We denote by $\mathbb{P}_{\mathrm{div}}(R)$ the set of all integer-valued Sylvester matrix rank functions on R. For each $\mathrm{rk} \in \mathbb{P}_{\mathrm{div}}(R)$, we say that the unique epic division R-ring $(\mathcal{D}, \varphi)$ in Corollary 3.1.17 is the *epic division envelope* of rk.

Remark 3.1.19. Note in particular that we have an embedding of R into a division ring if and only if there exists an integer-valued Sylvester matrix rank function rk on R with $\mathrm{rk}(r) = 1$ for every non-zero $r \in R$. Indeed, if $(\mathcal{D}, \varphi)$ is the epic division envelope of rk, then $\mathrm{rk}(r) = \mathrm{rk}_{\mathcal{D}}(\varphi(r)) = 1$ if and only if $\varphi(r)$ is invertible if and only if $\varphi(r)$ is non-zero. Thus, φ is injective if and only if $\mathrm{rk}(r) = 1$ for every non-zero $r \in R$.

Observe also that the use of the term "envelope" here is consistent with the corresponding use in Definition 1.4.5 since $\mathrm{rk}_{\mathcal{D}}$ is faithful. Therefore, $(\mathcal{D}, \varphi, \mathrm{rk}_{\mathcal{D}})$ is actually a regular envelope of rk, and since $\mathrm{rk}_{\mathcal{D}}$ is the unique Sylvester matrix rank function on $\mathcal{D}$ by Example 2.1.15(1.), including $\mathrm{rk}_{\mathcal{D}}$ in the triple is redundant. □

The next proposition shows the behavior of the natural transcendental extension of an integer-valued Sylvester matrix rank function.

Proposition 3.1.20. *Let R be a ring, τ an automorphism of R and* rk *an integer-valued τ-compatible Sylvester matrix rank function on R with epic division envelope $(\mathcal{D}, \varphi)$. Then,*

1.) *There exists an automorphism $\tilde{\tau}$ of $\mathcal{D}$ such that $\tilde{\tau} \circ \varphi = \varphi \circ \tau$ and $\mathrm{rk}_{\mathcal{D}}$ is $\tilde{\tau}$-compatible. In particular, this induces an epic ring homomorphism $\tilde{\varphi} : R[t^{\pm 1}; \tau] \to \mathcal{D}[t^{\pm 1}; \tilde{\tau}]$ that extends φ.*

2.) *If $\widetilde{\mathrm{rk}}_{\mathcal{D}}$ is the natural extension of $\mathrm{rk}_{\mathcal{D}}$ to $\mathcal{D}[t^{\pm 1}; \tilde{\tau}]$ then $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}_{\mathcal{D}})$ is the natural extension of* rk *to $R[t^{\pm 1}; \tau]$.*

3.) $\widetilde{\mathrm{rk}}$ *is integer-valued and its epic division envelope is isomorphic to the Ore division ring* $\mathcal{D}(t;\tilde{\tau})$ *of* $\mathcal{D}[t^{\pm 1};\tilde{\tau}]$.

Proof. Observe that, since $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}_{\mathcal{D}})$ is τ-compatible, we have for every matrix A over R that

$$\mathrm{rk}_{\mathcal{D}}(\varphi(A)) = \mathrm{rk}(A) = \mathrm{rk}(\tau(A)) = \mathrm{rk}_{\mathcal{D}}(\varphi(\tau(A))).$$

Since τ is an automorphism, $\varphi\circ\tau$ is a ring homomorphism $R \to \mathcal{D}$ whose image $\varphi\circ\tau(R) = \varphi(R)$ generates $\mathcal{D}$, i.e., $(\mathcal{D}, \varphi\circ\tau)$ is an epic division R-ring, and we have just proved that $(\varphi\circ\tau)^\sharp(\mathrm{rk}_{\mathcal{D}}) = \varphi^\sharp(\mathrm{rk}_{\mathcal{D}})$. By Corollary 3.1.15, there exists an automorphism $\tilde{\tau} : \mathcal{D} \to \mathcal{D}$ such that the following commutes

$$\begin{tikzcd} & R \arrow[dl, "\varphi"'] \arrow[dr, "\varphi\circ\tau"] & \\ \mathcal{D} \arrow[rr, "\tilde{\tau}"] & & \mathcal{D} \end{tikzcd}$$

Since in a division ring there exists only one Sylvester matrix rank function (see Example 2.1.15(1.)), necessarily $\tilde{\tau}^\sharp(\mathrm{rk}_{\mathcal{D}}) = \mathrm{rk}_{\mathcal{D}}$, i.e., $\mathrm{rk}_{\mathcal{D}}$ is $\tilde{\tau}$-compatible. From the commutativity of the previous diagram, the map $\tilde{\varphi} : R[t;\tau] \to \mathcal{D}[t;\tilde{\tau}]$ sending $r \mapsto \varphi(r)$ and $t \mapsto t$ defines a ring homomorphism that extends φ, and that can be further extended to $\tilde{\varphi} : R[t^{\pm 1};\tau] \to \mathcal{D}[t^{\pm 1};\tilde{\tau}]$. In addition, the map $\tilde{\varphi}$ is epic. Indeed, assume that we have two ring homomorphisms $f, g : \mathcal{D}[t^{\pm 1};\tilde{\tau}] \to Q$ to some ring Q such that $f\circ\tilde{\varphi} = g\circ\tilde{\varphi}$. In particular, $f(t) = f\circ\tilde{\varphi}(t) = g\circ\tilde{\varphi}(t) = g(t)$, and since the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook, "\iota_1"'] & \mathcal{D} \arrow[d, hook, "\iota_2"] \\ R[t^{\pm 1};\tau] \arrow[r, "\tilde{\varphi}"] & \mathcal{D}[t^{\pm 1};\tilde{\tau}] \end{tikzcd}$$

is commutative, we also deduce from $f\circ\tilde{\varphi} = g\circ\tilde{\varphi}$ that

$$f\circ\iota_2\circ\varphi = f\circ\tilde{\varphi}\circ\iota_1 = g\circ\tilde{\varphi}\circ\iota_1 = g\circ\iota_2\circ\varphi,$$

and therefore, since φ is epic, $f\circ\iota_2 = g\circ\iota_2$. Since f and g coincide on the indeterminate and on the elements of $\mathcal{D}$, $f = g$.

Now, since $\mathrm{rk}_{\mathcal{D}}$ is $\tilde{\tau}$-compatible and $\mathcal{D}$ is regular, we know that the Sylvester module rank function $\widetilde{\dim}_{\mathcal{D}}$ associated to the natural transcendental extension $\widetilde{\mathrm{rk}}_{\mathcal{D}}$ satisfies Proposition 1.5.5. Let us consider the maps $\psi_n : R[t;\tau] \to \mathrm{Mat}_n(R)$ and $\tilde{\psi}_n : \mathcal{D}[t;\tilde{\tau}] \to \mathrm{Mat}_n(\mathcal{D})$ defining the n^{th} extension $\widetilde{\mathrm{rk}}_n$ of rk to $R[t;\tau]$ and $\widetilde{\mathrm{rk}}_{\mathcal{D},n}$ of $\mathrm{rk}_{\mathcal{D}}$ to $\mathcal{D}[t;\tilde{\tau}]$, as in Eq. (1.1). If we build them from the corresponding canonical bases, we can see from the expression Eq. (1.2) and the relation $\tilde{\tau}\circ\varphi = \varphi\circ\tau$ that we have a commutative diagram

$$\begin{tikzcd} R[t;\tau] \arrow[r, "\psi_n"] \arrow[d, "\tilde{\varphi}"'] & \mathrm{Mat}_n(R) \arrow[d, "\varphi"] \\ \mathcal{D}[t;\tilde{\tau}] \arrow[r, "\tilde{\psi}_n"] & \mathrm{Mat}_n(\mathcal{D}) \end{tikzcd}$$

In particular, for every matrix A over R,

$$\widetilde{\mathrm{rk}}_n(A) = \frac{\mathrm{rk}(\psi_n(A))}{n} = \frac{\mathrm{rk}_{\mathcal{D}}(\varphi(\psi_n(A)))}{n} = \frac{\mathrm{rk}_{\mathcal{D}}(\widetilde{\psi_n}(\tilde{\varphi}(A)))}{n} = \widetilde{\mathrm{rk}}_{\mathcal{D},n}(\tilde{\varphi}(A)),$$

and hence

$$\widetilde{\mathrm{rk}}_{\mathcal{D}}(\tilde{\varphi}(A)) = \lim_{i\to\infty} \widetilde{\mathrm{rk}}_{\mathcal{D},i}(\tilde{\varphi}(A)) = \lim_{i\to\infty} \widetilde{\mathrm{rk}}_n(A),$$

i.e., the limit (exists and) equals $\widetilde{\mathrm{rk}}_{\mathcal{D}}(\tilde{\varphi}(A))$ for every A, what means precisely that $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}_{\mathcal{D}})$ as Sylvester matrix rank functions on $R[t;\tau]$. Given the way they are extended to skew Laurent polynomials, $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}_{\mathcal{D}})$ as Sylvester matrix rank functions on $R[t^{\pm 1};\tau]$.

Finally, for the last statement take any non-zero $p \in \mathcal{D}[t^{\pm 1};\tau]$ and express it as $p = t^{-k}q$ where $q \in \mathcal{D}[t;\tau]$ has non-zero constant term a_0. Then, since $\widetilde{\dim}_{\mathcal{D}}$ is a normalized length function (in particular, additive on short exact sequences), we obtain from Proposition 1.5.5 that

$$\widetilde{\mathrm{rk}}_{\mathcal{D}}(p) = \widetilde{\mathrm{rk}}_{\mathcal{D}}(q) = \widetilde{\dim}_{\mathcal{D}}(\mathcal{D}[t^{\pm 1};\tau]q) \geq \mathrm{rk}_{\mathcal{D}}(a_0) = 1.$$

Since $\mathcal{D}[t^{\pm 1};\tau]$ is an Ore domain (see Example 3.1.7), and every non-zero element has rank 1, we have by Proposition 2.1.9 that $\widetilde{\mathrm{rk}}_{\mathcal{D}}$ comes from its Ore division ring $\mathcal{D}(t;\tau)$. If $\iota : \mathcal{D}[t^{\pm 1};\tau] \to \mathcal{D}(t;\tau)$ denotes the natural embedding, then since the only Sylvester matrix rank on $\mathcal{D}(t;\tau)$ is $\mathrm{rk}_{\mathcal{D}(t;\tau)}$, we have that $\widetilde{\mathrm{rk}}_{\mathcal{D}} = \iota^\sharp(\mathrm{rk}_{\mathcal{D}(t;\tau)})$ is integer-valued. Since $\tilde{\varphi}$ and ι are epic (see Remark 3.1.6), so is their composition, and since $\widetilde{\mathrm{rk}} = (\tilde{\varphi})^\sharp\iota^\sharp(\mathrm{rk}_{\mathcal{D}(t;\tau)}) = (\iota\circ\tilde{\varphi})^\sharp(\mathrm{rk}_{\mathcal{D}(t;\tau)})$, $(\mathcal{D}(t;\tau), \iota\circ\tilde{\varphi})$ is the epic division envelope of $\widetilde{\mathrm{rk}}$. $\square$

Assume that we have somehow managed to prove that the ring R admits a homomorphism to a division ring, and observe that we can further assume that the homomorphism is epic by restricting to the division ring generated by the image of R (i.e. its division closure, as we introduce later in Section 3.3). We can ask whether, among all the possible epic division R-rings, there exists one in which we can invert "the most matrices possible". If the answer is positive, we call it the universal epic division R-ring, and if moreover it is a division R-ring of fractions, then we call it the universal division R-ring of fractions. Here we are only going to be interested in the latter object.

Definition 3.1.21. Let R be a ring. The division R-ring of fractions $(\mathcal{D}, \varphi)$ is called *the universal division R-ring of fractions* if for any other epic division R-ring $(\mathcal{D}', \varphi')$, the set Σ' of matrices that become invertible over $\mathcal{D}'$ is contained in the set Σ of matrices that become invertible over $\mathcal{D}$.

Observe from Theorem 3.1.14 *b*) that the universal division R-ring of fractions, if it exists, is unique up to R-isomorphism. In fact, the notion of universal epic division R-ring (or universal R-field) was introduced by P.M. Cohn (cf. [Coh06, Page 421]) as an initial object U in the category of epic division R-rings and specializations that we do not treat here, adding the postscript "of fractions" if R embeds in U. From here, the equivalence with the previous definition can be obtained through [Coh06, Theorem 7.2.7].

The definition of the universal division R-ring of fractions can be restated in terms of Sylvester matrix rank functions, in a way similar to Corollary 3.1.15. Let us first define universal rank functions.

Definition 3.1.22.

i) Given two Sylvester matrix rank functions rk_1 and rk_2 on a ring R, we write $\mathrm{rk}_1 \leq \mathrm{rk}_2$ if, for every matrix A over R, we have $\mathrm{rk}_1(A) \leq \mathrm{rk}_2(A)$.

ii) Given a family $\mathcal{F} \subseteq \mathbb{P}(R)$ of Sylvester matrix rank functions, we say that $\mathrm{rk} \in \mathcal{F}$ is *maximal in* $\mathcal{F}$ if there exists no $\mathrm{rk}' \in \mathcal{F} \setminus \{\mathrm{rk}\}$ with $\mathrm{rk} \leq \mathrm{rk}'$, and we say that it is *universal in* $\mathcal{F}$ if $\mathrm{rk}' \leq \mathrm{rk}$ for any other $\mathrm{rk}' \in \mathcal{F}$.

With these definitions, the universal division R-ring of fractions is characterized as follows.

Proposition 3.1.23. *Let R be a ring. The division R-ring of fractions $(\mathcal{D}, \varphi)$ is universal if and only if $\varphi^\sharp(\mathrm{rk}_\mathcal{D})$ is universal in $\mathbb{P}_{\mathrm{div}}(R)$, i.e., if for any other epic division R-ring $(\mathcal{D}', \varphi')$, we have $(\varphi')^\sharp(\mathrm{rk}_{\mathcal{D}'}) \leq \varphi^\sharp(\mathrm{rk}_\mathcal{D})$.*

Proof. Note that the restatement of universality in $\mathbb{P}_{\mathrm{div}}(R)$ follows from the bijection described in Theorem 3.1.16. Now, we proved in Proposition 1.1.14 4. that the rank of a matrix over a division ring equals the size of its biggest invertible square submatrix.

Assume first that $(\mathcal{D}, \varphi)$ is universal, let $(\mathcal{D}', \varphi')$ be another epic division R-ring and take a matrix A over R. If $\mathrm{rk}_{\mathcal{D}'}(\varphi'(A)) = k$, then there exists a $k \times k$ invertible submatrix of $\varphi'(A)$, which is then of the form $\varphi'(A_0)$ for some $k \times k$ submatrix A_0 of A. Since $(\mathcal{D}, \varphi)$ is universal, $\varphi(A_0)$ is a $k \times k$ invertible submatrix of $\varphi(A)$, and therefore $\mathrm{rk}_{\mathcal{D}'}(\varphi'(A)) = k \leq \mathrm{rk}_\mathcal{D}(\varphi(A))$. Thus, $(\varphi')^\sharp(\mathrm{rk}_{\mathcal{D}'}) \leq \varphi^\sharp(\mathrm{rk}_\mathcal{D})$.

Conversely, if $\varphi^\sharp(\mathrm{rk}_\mathcal{D}) \geq (\varphi')^\sharp(\mathrm{rk}_{\mathcal{D}'})$ for every epic division R-ring $(\mathcal{D}', \varphi')$ and A is an $n \times n$ matrix over R such that $\varphi'(A)$ is invertible over $\mathcal{D}'$, then $\mathrm{rk}_\mathcal{D}(\varphi(A)) \geq \mathrm{rk}_{\mathcal{D}'}(\varphi'(A)) = n$, and hence $\varphi(A)$ is invertible over $\mathcal{D}$. Therefore, $(\mathcal{D}, \varphi)$ is universal. □

In Section 3.2 we recall that Sylvester and pseudo-Sylvester domains admit a universal division ring of fractions for which the set Σ of matrices becoming invertible under the embedding can be characterized in a natural way in terms of the inner and the stable rank, respectively.

In the recent preprint [KS20], the authors develop the analog of Cohn's theory of epic division rings in the context of group graded rings and group graded division rings.

3.2 Sylvester and pseudo-Sylvester domains

Recall that we introduced Sylvester and pseudo-Sylvester domains in Section 1.1 as the families of stably finite rings satisfying Sylvester's law of nullity with respect to the inner and stable rank, respectively. These two families of rings can also be characterized in terms of the universal division rings of fractions. Since we deal with homomorphisms of rings, in the following we use ρ_R (resp. ρ_R^*) to denote the inner (resp. stable) rank over the ring R.

Let R be a ring and let $(\mathcal{D}, \varphi)$ be an epic division R-ring. Observe that, given an $n \times n$ matrix A over R, a clear obstruction for the invertibility of $\varphi(A)$ in $\mathcal{D}$ is that A is not stably full. Indeed, if A is not stably full, then there exists $s \geq 0$ such that $\rho_R(A \oplus I_s) < n + s$, so we can express $A \oplus I_s = PQ$ where P (resp. Q) is a matrix over R with less than $n + s$ columns (resp. rows). This leads to the corresponding decomposition of $\varphi(A \oplus I_s)$ in $\mathcal{D}$, what implies that $\mathrm{rk}_{\mathcal{D}}(\varphi(A \oplus I_s)) < n + s$. Therefore, $\varphi(A \oplus I_s) = \varphi(A) \oplus I_s$, and consequently $\varphi(A)$, is not invertible in $\mathcal{D}$.

Thus, we can naively wonder whether this is the only obstruction for the invertibility of a square matrix, i.e., whether there exists an epic division R-ring $(\mathcal{D}, \varphi)$ in which every stably full matrix over R can be inverted, which would then clearly be the universal division R-ring. Moreover, this universal division R-ring would be the universal division R-ring of fractions if and only if R is stably finite. Indeed, if R is stably finite, then in particular every non-zero element of R is stably full (see Proposition 1.1.5), and hence become invertible in $\mathcal{D}$, so in particular φ must be injective. Conversely, if φ is injective, then R, as a subring of a division ring, would be stably finite.

The family of rings that admit such a universal division R-ring of fractions is precisely the family of pseudo-Sylvester domains introduced in [CS82], as we will state in a moment. In this sense, since in a Sylvester domain every full matrix is actually stably full (see Corollary 1.1.13), Sylvester domains form the family of rings admitting a universal division R-ring of fractions in which all full matrices become invertible.

The characterization of Sylvester and pseudo-Sylvester domains given here corresponds to [Coh06, Theorem 7.5.13 & Theorem 7.5.18] but in a compressed form. We do not present a new proof of this fact, but since the absence of the "rank preserving" property may make the result look weaker, we add a few lines using the previously cited theorems to clarify the equivalence. The bracketed statements correspond to the result for pseudo-Sylvester domains.

Theorem 3.2.1. *For a non-zero ring R, the following are equivalent:*

i) R is a (pseudo)-Sylvester domain.

ii) There exists a division R-ring of fractions $R \hookrightarrow \mathcal{D}$ in which every (stably) full matrix over R becomes invertible.

Moreover, if R satisfies one, and hence each of the previous properties, $\mathcal{D}$ is the universal division R-ring of fractions, and it is isomorphic to the universal localization of R with respect to the set of all (stably) full matrices over R.

Proof. Since for a Sylvester domain every full matrix is stably full as discussed earlier, it suffices to work with the latter. Let Σ be the set of stably full matrices over R.

Assume (1). Then [Coh06, Theorem 7.5.13 (e)] (resp. [Coh06, Theorem 7.5.18 (c)] and its conclusion) tells us that R_Σ is a division R-ring of fractions in which, by definition, every stably full matrix becomes invertible (thus, it is the universal one).

Assume (2). Then R, as a subring of a division ring, is stably finite. Moreover, since a matrix over R which is not stably full cannot be inverted over any division ring, Σ is precisely the set of matrices becoming invertible over $\mathcal{D}$ and it follows from [Coh06, Theorem 7.4.3(ii)] that its complement in the set of square matrices is a prime matrix ideal, which implies (1) by [Coh06, Theorem 7.5.13(d)] (resp. [Coh06, Theorem 7.5.18(b)]). $\square$

Sylvester and pseudo-Sylvester domains also enjoy particular homological properties. For instance, Sylvester domains are projective-free, while on pseudo-Sylvester domains every finitely generated projective module is stably free. Recall the following.

Definition 3.2.2. Let R be a ring. A left or right R-module P is said to be *stably free* if there exists $n \geq 0$ such that $P \oplus R^n$ is a free R-module.

In particular, observe that stably free modules are projective. By a result of Gabel, a proof of which is given in [Lam78, Proposition 4.2], any stably free module that is not finitely generated is already free, and hence we can restrict our attention to finitely generated stably free modules.

Now, if P is finitely generated stably free and $P \oplus R^n$ is free, then this free module is necessarily finitely generated and hence isomorphic to some R^m. In general, the difference $m - n$ need neither be positive nor uniquely determined by P. Nevertheless, it is unique and positive for a non-zero module if R is stably finite by Remark 1.1.3. Indeed, if P is non-trivial and we have $n \geq m$, then from the isomorphism $(P \oplus R^{n-m}) \oplus R^m \cong R^m$ we deduce that $P \oplus R^{n-m} = 0$, a contradiction. Therefore, we always have $m > n$. Similarly, if we have two expressions $P \oplus R^n \cong R^m$ and $P \oplus R^k \cong R^l$ with $k \geq n$, then

$$R^l \cong P \oplus R^n \oplus R^{k-n} \cong R^m \oplus R^{k-n} \cong R^{m+k-n},$$

and therefore $m + k - n = l$, i.e., $m - n = l - k$. In this case, this positive constant will be denoted $\mathrm{rk}_{sf}(P)$ and will be called *the rank of the stably free module P*.

Definition 3.2.3. Let R be a ring.

a) We say that R is *projective-free* if every finitely generated projective R-module is free of unique rank.

b) If R is stably finite, then we say that R has *stably free cancellation* (*SFC*) if every finitely generated stably free R-module P is free of rank $\mathrm{rk}_{sf}(P)$.

In [Coh06], rings with invariant basis number (IBN) and stably free cancellation are called *Hermite rings*. Here, we keep the terminology "stably free cancellation" because Hermite rings have also been given other meanings in the literature.

Note that in the definitions we have not made explicit a choice of sides (left or right) for modules. This is due to the following result.

Lemma 3.2.4. *Let R be a ring. The following statements hold for left modules if and only if they hold for right modules.*

a) Every finitely generated projective R-module is free.

b) Every finitely generated projective R-module is stably free.

c) Every finitely generated stably free R-module is free.

If R is stably finite, the rank of the free R-module in (a) and (c), and of the stably free module in (b), is unique.

Proof. To every left R-module M we can associate the dual module $M^* = \mathrm{Hom}_R(M, {}_RR)$, which is a right R-module with fr defined as $(fr)(x) = f(x)r$ for every $r \in R$ and $x \in M$. Analogously, the dual of a right R-module N is the left R-module $N^* = \mathrm{Hom}_R(N, R_R)$ with rg defined by $(rg)(y) = rg(y)$ for every $r \in R$ and $y \in N$ (cf. [Rot09, Proposition 2.54 (iii) and (iv)]).

Since $\mathrm{Hom}_R({}_RR, {}_RR) \cong R_R$, $\mathrm{Hom}_R(R_R, R_R) \cong {}_RR$ and the contravariant Hom functors commute with finite direct sums (cf. [Rot09, Corollary 2.32]), we have that whenever M or N are finitely generated free of rank n (resp. projective and a direct summand of R^n), so are their duals. Furthermore, in these two cases we have an isomorphim of left R-modules $M \cong M^{**}$ and of right R-modules $N \cong N^{**}$ (cf. [Lam99, Corollary 2.10 & Remark 2.11]).

Assume then that statement *a)* holds for every finitely generated projective left R-module, and let P be a finitely generated projective right R-module. By the previous discussion, P^* is a finitely generated projective left R-module, and hence free, say $P^* \cong {}_RR^n$. Thus, $P \cong P^{**} \cong ({}_RR^n)^* = R_R^n$ is free of the same rank. Similarly if *b)* holds we would have $P^* \oplus {}_RR^n \cong {}_RR^m$ and consequently $R_R^m \cong ({}_RR^m)^* = (P^* \oplus {}_RR^n)^* \cong P^{**} \oplus R_R^n \cong P \oplus R_R^n$, so P is stably free. Finally, if *c)* holds and $P \oplus R_R^n \cong R_R^m$, then from the isomorphism $P^* \oplus {}_RR^n \cong {}_RR^m$ we have that P^* is free, say $P^* \cong {}_RR^k$, and therefore $P \cong P^{**} \cong R_R^k$ is free of the same rank. □

Therefore, to check that a ring R is projective-free or has stably free cancellation, it suffices to check it for left R-modules. As we anticipated before, one has the following for Sylvester (cf. [DS78, Theorem 6 & the subsequent discussion], or [Coh06, Corollary 5.5.7]) and pseudo-Sylvester domains (cf.[Coh06, Proposition 5.6.2]).

Proposition 3.2.5. *Every Sylvester domain is projective-free, and on a pseudo-Sylvester domain, every finitely generated projective module is stably free.*

Moreover, it can be shown that a pseudo-Sylvester domain is a Sylvester domain if and only if it admits stably free cancellation ([CS82, Proposition 6.1], or [Coh06, Proposition 5.6.1]).

Recall that if K is a field, then the polynomial ring $K[t_1, \ldots, t_n]$ in n indeterminates is projective-free, a result known as the Quillen-Suslin Theorem. If $n = 2$, then the polynomial ring is in fact a Sylvester domain, but this is no longer true for $n \geq 3$ (cf. [Coh06, Corollary 5.5.5 and the subsequent discussion]), so these are examples of domains that are projective-free but not Sylvester domains.

Another homological feature of Sylvester domains has to do with flat resolutions of modules. We use the opportunity to introduce also projective resolutions and their relations with the functors Tor and Ext, since we need this later in Section 3.5.

Definition 3.2.6. An R-module M has *projective dimension* at most n (abbreviated $\mathrm{pd}(M) \leq n$) if M admits a resolution

$$0 \to P_n \to \ldots \to P_0 \to M \to 0$$

of projective R-modules. The supremum among the projective dimensions of all left (resp. right) R-modules is called the *left* (resp. *right*) *global dimension* of R.

Remark. The left and right global dimensions of a ring do not necessarily coincide in general (cf. [Jat69]). □

Observe in particular from the definition that M is projective if and only if $\mathrm{pd}(M) = 0$. The following lemma, which corresponds to [Rot09, Proposition 8.6], shows that this concept is deeply related to Ext functors. Although we state the result for left R-modules, the same holds for right R-modules.

Lemma 3.2.7. *The following are equivalent for a left R-module N:*

1. *The projective dimension of N is at most n.*
2. $\mathrm{Ext}_R^i(N, N') = 0$ *for all left R-modules N' and $i > n$.*
3. $\mathrm{Ext}_R^{n+1}(N, N') = 0$ *for all left R-modules N'.*
4. *If $0 \to I \to P_{n-1} \to \ldots P_0 \to N \to 0$ is an exact sequence where every P_i is projective, then I is projective.*

Analogously, we can define the flat dimension of a module in terms of flat resolutions.

Definition 3.2.8. An R-module M has *flat dimension* at most n (abbreviated $\mathrm{fd}(M) \leq n$), if it admits a resolution of flat R-modules

$$0 \to Q_n \to \ldots \to Q_0 \to M \to 0.$$

We define the *left* (resp. *right*) *weak dimension* of R as the supremum of the flat dimensions of all left (resp. right) R-modules. It turns out that this notion is always left-right symmetric (cf. [Rot09, Theorem 8.19]) and hence we can just talk about the *weak dimension* of R.

Again, an R-module M is flat if and only if $\mathrm{fd}(M) = 0$ and, as it happens with the projective dimension and the Ext functor, the flat dimension of a left R-module N (resp. of a right R-module M) can be characterized in terms of $\mathrm{Tor}_*^R(\square, N)$ (resp. $\mathrm{Tor}_*^R(M, \square)$). Observe though that, unlike the previous case, here we need to change the argument while considering left or right modules. This lemma corresponds to the left version of [Rot09, Proposition 8.17].

Lemma 3.2.9. *The following are equivalent for a left R-module N:*

1. *The flat dimension of N is at most n.*

2. $\operatorname{Tor}_i^R(M, N) = 0$ *for all right R-modules M and $i > n$.*

3. $\operatorname{Tor}_{n+1}^R(M, N) = 0$ *for all right R-modules M.*

4. *If $0 \to J \to Q_{n-1} \to \dots Q_0 \to N \to 0$ is an exact sequence where every Q_i is flat, then J is flat.*

As we already mentioned above, we have the following (cf. [DS78, Theorem 6] or [Coh06, Corollary 5.5.7], and [CS82, Theorem 3.4 & Section 6]).

Proposition 3.2.10. *A (pseudo-)Sylvester domain has weak dimension at most 2.*

We are now ready to state the homological recognition principles for Sylvester and pseudo-Sylvester domains. In [Jai20C, Proposition 2.2 & Theorem 2.4], A. Jaikin-Zapirain provided the following homological characterization of Sylvester domains. Given a matrix A over R, we consistently denote by $\operatorname{rk}_{\mathcal{D}}(A)$ the rank of A considered as a matrix over $\mathcal{D}$, and similarly, if M is a left (resp. right) R-module, we take $\dim_{\mathcal{D}}(M)$ to denote $\dim_{\mathcal{D}}(\mathcal{D} \otimes_R M)$ (resp. $\dim_{\mathcal{D}}(M \otimes_R \mathcal{D})$).

Theorem 3.2.11. *A ring R is a Sylvester domain if and only if there exists a division R-ring of fractions $R \hookrightarrow \mathcal{D}$ such that:*

(1) $\operatorname{Tor}_1^R(\mathcal{D}, \mathcal{D}) = 0$.

(2) For any finitely generated left or right R-submodule M of $\mathcal{D}$ and any exact sequence $0 \to J \to R^n \to M \to 0$, J is a direct union of submodules isomorphic to R^k, where $k = \dim_{\mathcal{D}}(J)$.

In this event, $\mathcal{D}$ is the universal division R-ring of fractions, which coincides with the universal localization of R at the set Σ of all full matrices.

As a particular corollary we obtain the following sufficient condition for a ring to be a Sylvester domain.

Corollary 3.2.12. *Let R be a ring and assume that there exists a division R-ring of fractions $R \hookrightarrow \mathcal{D}$ such that*

(1) $\operatorname{Tor}_1^R(\mathcal{D}, \mathcal{D}) = 0$ *and*

(2) For any finitely generated left or right R-submodule M of $\mathcal{D}$ and any exact sequence $0 \to J \to R^n \to M \to 0$, the R-module J is free of finite rank.

Then R is a Sylvester domain and $\mathcal{D}$ is the universal division R-ring of fractions, hence (isomorphic to) the universal localization of R at the set Σ of all full matrices.

We build on our homological recognition principle for pseudo-Sylvester domains based on the previous corollary, and we obtain then the following.

Theorem 3.2.13. *Let $R \hookrightarrow \mathcal{D}$ be an epic division R-ring. Assume that*

1. $\mathrm{Tor}_1^R(\mathcal{D},\mathcal{D}) = 0$ *and*

2. for any finitely generated left or right R-submodule M of $\mathcal{D}$ and any exact sequence $0 \to J \to R^n \to M \to 0$, the R-module J is finitely generated stably free.

Then R is a pseudo-Sylvester domain and $\mathcal{D}$ is the universal division R-ring of fractions, hence (isomorphic to) the universal localization of R with respect to all stably full matrices.

Proof. Notice that by Theorem 3.2.1 it suffices to show that every stably full matrix over R becomes invertible over $\mathcal{D}$. Thus, let A be an $n \times n$ matrix over R with $\rho^*(A) = n$, and assume that A is not invertible over $\mathcal{D}$, i.e., $\mathrm{rk}_{\mathcal{D}}(A) < n$. Since R is a subring of a division ring, it is necessarily stably finite.

Let N be the left R-module $N = R^n/R^nA$. Then A is also the presentation matrix of $\mathcal{D} \otimes_R N$, and therefore $\dim_{\mathcal{D}}(N) = n - \mathrm{rk}_{\mathcal{D}}(A)$, which is finite and positive. This implies that $\mathcal{D} \otimes_R N \cong \mathcal{D}^k$ as $\mathcal{D}$-modules for some $k \geq 1$ and, thus, composing the R-homomorphism $N \to \mathcal{D} \otimes_R N$ given by $x \to 1 \otimes x$ with an appropriate projection, we obtain a non-trivial R-homomorphism $N \to \mathcal{D}$. Therefore, if M is the image of this map, the surjection $N \to M$ gives us a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R^nA \arrow[r] \arrow[d, dotted] & R^n \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & R^n \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Here, J is the kernel of the map $R^n \to M$ and the dotted arrow is such that the left square commutes (cf. [Rot09, Proposition 2.71]) and therefore injective. Moreover, notice that $\mathcal{D} \otimes_R M$ is non-trivial since the multiplication map to $\mathcal{D}$ is non-trivial. We conclude that $\dim_{\mathcal{D}}(M) > 0$.

Now we have by (2) that J is stably free, i.e., there exists $s \geq 0$ such that $J \oplus R^s$ is free. Moreover, since J is finitely generated and R, as a subring of a division ring, is stably finite, we conclude that $J \oplus R^s \cong R^{\mathrm{rk}_{sf}(J)+s}$. In fact, we obtain that $\mathrm{rk}_{sf}(J) = \dim_{\mathcal{D}}(J)$ by applying $\mathcal{D} \otimes_R \square$. Notice also that the previous diagram remains exact and commutative if we add $0 \to R^s \to R^s \to 0 \to 0$ to both rows. Thus, setting $t := \dim_{\mathcal{D}}(J)$, the situation can be summarized in the following commutative diagram:

$$\begin{tikzcd}
R^{n+s} \arrow[d, "r_{A\oplus I_s}"'] \arrow[dr, "r_{A\oplus I_s}"] & \\
R^nA \oplus R^s \arrow[r, hook] \arrow[d, hook] & R^{n+s} \arrow[d, equal] \\
R^{t+s} \cong J \oplus R^s \arrow[r, hook] & R^{n+s}.
\end{tikzcd}$$

Here, $r_{A\oplus I_s}$ denotes the homomorphism given by right multiplication by $A \oplus I_s$, so that all maps except the isomorphism behave identically on the R^s summand. In terms of matrices, this factorization of $r_{A\oplus I_s}$ allows us to express $A \oplus I_s$ as a product of two matrices of dimensions $(n+s)\times(t+s)$ and $(t+s)\times(n+s)$, respectively. Thus, $\rho(A\oplus I_s)\le t+s$ right by definition. Since A is stably full, we have $\rho(A\oplus I_s)=n+s$ for every s, so we conclude that $n\le t$.

We are going to show on the other hand that $t<n$, a contradiction. Observe first that the condition (2) tells us in particular that the flat (in fact, projective) dimension of any finitely generated right R-submodule of $\mathcal{D}$ is at most 1. Hence, using Lemma 3.2.9 and the fact that Tor commutes with directed colimits (cf. [Rot09, Proposition 7.8]), we obtain that for any left R-module Q,

$$\mathrm{Tor}_2^R(\mathcal{D},Q)=\mathrm{Tor}_2^R\left(\varinjlim L_i,Q\right)\cong\varinjlim \mathrm{Tor}_2^R(L_i,Q)=0,$$

where L_i runs through all finitely generated R-submodules of the right R-module $\mathcal{D}$. Again by Lemma 3.2.9, this means that $\mathcal{D}$ itself has flat dimension at most 1 as a right R-module.

Now, since M is an R-submodule of $\mathcal{D}$, we have an exact sequence of left R-modules $0\to M\to\mathcal{D}\to Q\to 0$ for some left R-module Q, and hence, applying $\mathcal{D}\otimes_R\square$ we can construct a long exact sequence containing the following exact part:

$$\cdots\to\mathrm{Tor}_2^R(\mathcal{D},Q)\to\mathrm{Tor}_1^R(\mathcal{D},M)\to\mathrm{Tor}_1^R(\mathcal{D},\mathcal{D})\to\cdots.$$

The first term is trivial by the previous argument, while the third term is trivial because of (1). Thus, we deduce that $\mathrm{Tor}_1^R(\mathcal{D},M)=0$. From here, it follows that applying $\mathcal{D}\otimes_R\square$ to the exact sequence $0\to J\to R^n\to M\to 0$ returns an exact sequence of left $\mathcal{D}$-modules

$$0\to\mathcal{D}\otimes_R J\to\mathcal{D}^n\to\mathcal{D}\otimes_R M\to 0,$$

from which we obtain

$$t=\dim_{\mathcal{D}}(J)=n-\dim_{\mathcal{D}}(M)<n.$$

This is the desired contradiction, which shows that necessarily $\mathrm{rk}_{\mathcal{D}}(A)=n$. $\square$

We use Corollary 3.2.12 and Theorem 3.2.13 in Section 3.5 to explore conditions under which a crossed product of the form $\mathfrak{F}*\mathbb{Z}$, where $\mathfrak{F}$ is a fir, is a (pseudo)-Sylvester domain, and we further give examples of rings of the previous form that are Sylvester domains, and examples that are pseudo-Sylvester domains but not Sylvester domains.

3.3 Division and rational closures

Up to now, given a ring R, we have always worked with epic division R-rings, so that every element in the latter can be constructed from the image of R by means of addition, substraction, multiplication and taking inverses. In general, given any homomorphism $\varphi:R\to\mathcal{D}$ to a division ring $\mathcal{D}$, the subring $\mathcal{D}_{\varphi(R),\mathcal{D}}$ of $\mathcal{D}$ constructed from $\varphi(R)$ using the previous operations is a division subring of $\mathcal{D}$, and hence the restriction $\varphi:R\to\mathcal{D}_{\varphi(R),\mathcal{D}}$ is epic. This ring, $\mathcal{D}_{\varphi(R),\mathcal{D}}$, which is usually called the *division closure of* $\varphi(R)$ *in* $\mathcal{D}$ (or simply the division closure of R in $\mathcal{D}$ if the map is understood from the context), is the smallest subring of $\mathcal{D}$ that contains $\varphi(R)$ and is closed under taking inverses. More generally, given any homomorphism $R \to S$, a subring of S with this description can always be constructed, as follows.

Definition 3.3.1. Let R be a subring of a ring S. The *division closure of R in S* is the smallest subring $\mathcal{D}_{R,S}$ of S that contains R and is closed under taking inverses, i.e., if $x \in \mathcal{D}_{R,S}$ and x is invertible over S, then $x^{-1} \in \mathcal{D}_{R,S}$. We say that R is *division closed in S* if $R = \mathcal{D}_{R,S}$.

Observe that, unlike the case where $S = \mathcal{D}$ is a division ring, the division closure is not necessarily a division subring of S, for if a non-zero $x \in R$ is not invertible in S, then it cannot be invertible in $\mathcal{D}_{R,S}$.

Proposition 3.3.2. *Let R be a subring of a ring S. The division closure $\mathcal{D}_{R,S}$ of R in S is the intersection of all subrings of S containing R and that are closed under taking inverses. Moreover, it can be constructed inductively as follows:*

1. *Set $Q_0 := R$.*

2. *Suppose $n \geq 0$ and that we have constructed a subring Q_n of S. Define Q_{n+1} to be the subring of S generated by the elements of Q_n and their inverses (whenever they exist).*

Then $\mathcal{D}_{R,S} = \bigcup_{n=0}^{\infty} Q_n$.

Proof. Let $\{S_i\}$ be the family of all subrings of S that contain R and that are closed under taking inverses. Then $\bigcap_i S_i$ contains R and, if $x \in \bigcap_i S_i$ is invertible over S, then by the assumption on each S_i, $x^{-1} \in \bigcap_i S_i$. Clearly $\bigcap_i S_i$ is the smallest subring with such properties, so that $\mathcal{D}_{R,S} = \bigcap_i S_i$.

On the other hand, since $Q_n \subseteq Q_{n+1}$ for every $n \geq 0$, the inductive construction $S_0 = \bigcup_{n=0}^{\infty} Q_n$ is a subring of S containing R and such that, if $x \in S_0$, say $x \in Q_{n_x}$, and x is invertible over S, then by construction $x^{-1} \in Q_{n_x+1} \subseteq S_0$. Moreover, if S' is another subring of S with the previous properties, then we can inductively see that $Q_n \subseteq S'$ for every $n \geq 0$, and therefore $S_0 \subseteq S'$. Thus, $S_0 = \mathcal{D}_{R,S}$. $\square$

The following are basic properties of division closures.

Lemma 3.3.3. *Let $T \subseteq R \subseteq S$ be rings. The following hold.*

(1) $\mathcal{D}_{T,S} = \mathcal{D}_{T,\mathcal{D}_{R,S}} \subseteq \mathcal{D}_{R,S}$.

(2) If $R = \mathcal{U}$ is regular, then $\mathcal{D}_{T,S} = \mathcal{D}_{T,\mathcal{U}}$. In particular, if $T = \mathcal{U}$, then $\mathcal{U} = \mathcal{D}_{\mathcal{U},S}$, i.e., $\mathcal{U}$ is division closed in every overring.

(3) If $\varphi : R \to R'$ is a ring isomorphism, then $\mathcal{D}_{T,R} \cong \mathcal{D}_{\varphi(T),R'}$ as T-rings.

Proof.
(1) $\mathcal{D}_{R,S}$ is a subring of S containing $R \supseteq T$ and that is closed under taking inverses, and hence $\mathcal{D}_{T,S} \subseteq \mathcal{D}_{R,S}$. Therefore, $\mathcal{D}_{T,S}$ is a subring of $\mathcal{D}_{R,S}$ containing T and satisfying that if $x \in \mathcal{D}_{T,S}$ is invertible over $\mathcal{D}_{R,S} \subseteq S$, then $x^{-1} \in \mathcal{D}_{T,S}$. Thus, $\mathcal{D}_{T,\mathcal{D}_{R,S}} \subseteq \mathcal{D}_{T,S} \subseteq \mathcal{D}_{R,S}$.

Conversely, $\mathcal{D}_{T,\mathcal{D}_{R,S}}$ is a subring of $\mathcal{D}_{R,S} \subseteq S$ containing T and satisfying that, if $x \in \mathcal{D}_{T,\mathcal{D}_{R,S}} \subseteq \mathcal{D}_{R,S}$ is invertible over S, then by definition $x^{-1} \in \mathcal{D}_{R,S}$, and again by definition $x^{-1} \in \mathcal{D}_{T,\mathcal{D}_{R,S}}$. Therefore, $\mathcal{D}_{T,S} \subseteq \mathcal{D}_{T,\mathcal{D}_{R,S}}$ and we have proved that

$$\mathcal{D}_{T,S} = \mathcal{D}_{T,\mathcal{D}_{R,S}} \subseteq \mathcal{D}_{R,S}$$

(2) Let us first prove that $\mathcal{U} = \mathcal{D}_{\mathcal{U},S}$. Indeed, if $x \in \mathcal{U}$ is invertible over S, then it cannot be a zero-divisor in $\mathcal{U}$. Therefore, if $y \in \mathcal{U}$ is such that $xyx = x$, we deduce that necessarily $yx = 1$ and $xy = 1$, so $x^{-1} = y \in \mathcal{U}$. Using this together with the equality in (1), we obtain that $\mathcal{D}_{T,S} = \mathcal{D}_{T,\mathcal{D}_{\mathcal{U},S}} = \mathcal{D}_{T,\mathcal{U}}$.

(3) Assume that we have decompositions $\mathcal{D}_{T,R} = \bigcup_{n=0}^{\infty} Q_n$ and $\mathcal{D}_{\varphi(T),R'} = \bigcup_{n=0}^{\infty} Q'_n$ as in Proposition 3.3.2. We show by induction that, for every $i \geq 0$, $\varphi(Q_i) = Q'_i$, from where it follows that $\varphi(\mathcal{D}_{T,R}) = \mathcal{D}_{\varphi(T),R'}$.

From their construction, $Q_0 = T$ and $Q'_0 = \varphi(T)$, so that $\varphi(Q_0) = Q'_0$. Assume that we have proved that $\varphi(Q_i) = Q'_i$ for some $i \geq 0$. If $x \in Q_i$ is invertible in R, then $x^{-1} \in Q_{i+1}$ and $\varphi(x^{-1}) = \varphi(x)^{-1} \in Q'_{i+1}$ by the induction hypothesis. Therefore, the image of every generator of Q_{i+1} is in Q'_{i+1}. Since the latter is a ring and φ is a ring homomorphism, we deduce that $\varphi(Q_{i+1}) \subseteq Q'_{i+1}$. Conversely, if $y \in Q'_i$ is invertible in R', then by the induction hypothesis $y = \varphi(x)$ for some $x \in Q_i$ and x is invertible because φ is an isomorphism. Thus, $y^{-1} = \varphi(x)^{-1} = \varphi(x^{-1}) \in \varphi(Q_{i+1})$. Hence, $\varphi(Q_{i+1})$ is a subring of R' containing every generator of Q'_{i+1}, and hence $Q'_{i+1} \subseteq \varphi(Q_{i+1})$. This gives us the desired equality.

Consequently, the restriction of φ to $\mathcal{D}_{T,R}$ is an isomorphism from $\mathcal{D}_{T,R}$ to $\mathcal{D}_{\varphi(T),R'}$, which is of T-rings by definition. $\square$

The next lemma will also be of interest in the next section, since it allows us to extend an automorphism of a ring R given by conjugation by an invertible element in the overring S to an automorphism of the division closure.

Lemma 3.3.4. *Let R be a subring of a ring S, and assume that there exists an invertible element $s \in S$ such that $sRs^{-1} = R$. Then $s\mathcal{D}_{R,S}s^{-1} = \mathcal{D}_{R,S}$.*

Proof. Recall from Proposition 3.3.2 that we can construct $\mathcal{D}_{R,S}$ by setting $Q_0 = R$, defining inductively Q_i for $i \geq 1$ as the subring of S generated by Q_{i-1} and the inverses in S of the elements of Q_{i-1}, and taking $\bigcup_{i=0}^{\infty} Q_i$. We have by hypothesis that $sQ_0s^{-1} = Q_0$. Assume by induction that we have proved that $sQ_{i-1}s^{-1} = Q_{i-1}$ for some $i \geq 1$. If $x \in Q_i$ is such that $x = y^{-1}$ for some $y \in Q_{i-1}$, then

$$sxs^{-1} = sy^{-1}s^{-1} = (sys^{-1})^{-1},$$

and since $sys^{-1} \in Q_{i-1}$ by the inductive hypothesis and it is invertible, its inverse sxs^{-1} belongs to Q_i. Therefore, we have seen that $sx's^{-1} \in Q_i$ for every generator of Q_i. Taking into account that if $r_1, r_2 \in Q_i$ satisfy $sr_1s^{-1}, sr_2s^{-1} \in Q_i$, then $s(r_1 \pm r_2)s^{-1} = sr_1s^{-1} \pm sr_2s^{-1} \in Q_i$ and $sr_1r_2s^{-1} = (sr_1s^{-1})(sr_2s^{-1}) \in Q_i$, we deduce that $sQ_is^{-1} \subseteq Q_i$. Reasoning similarly, we can deduce that $s^{-1}Q_is \subseteq Q_i$, and hence $sQ_is^{-1} = Q_i$. Since this holds for every i, we finally obtain that $s\mathcal{D}_{R,S}s^{-1} = \mathcal{D}_{R,S}$. □

There is another notion of closure which is deeply related to Σ-inverting homomorphisms.

Definition 3.3.5. Let $\varphi : R \to S$ be a ring homomorphism and let Σ_φ be the set of all matrices over R whose image is invertible over S. The *rational closure of R in S*, denoted $R^\varphi(S)$, is the subset of S consisting of all entries of the matrices $\varphi(A)^{-1}$ for $A \in \Sigma_\varphi$, i.e.,

$$R^\varphi(S) = \{a'_{ij} : \exists A \in \Sigma_\varphi \text{ such that } \varphi(A)^{-1} = (a'_{ij})\}.$$

Note that, by definition, the homomorphism $\varphi : R \to S$ is Σ_φ-inverting. Observe also that $R^\varphi(S)$ contains $\varphi(R)$ since for every $r \in R$, $\varphi(r)$ is an entry of the inverse of

$$\begin{pmatrix} 1 & -\varphi(r) \\ 0 & 1 \end{pmatrix} = \varphi\begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \in \varphi(\Sigma_\varphi).$$

In the terminology of [Coh06, Section 7.1], the set Σ_φ is lower and upper multiplicative, i.e., it satisfies:

(*i*) $1 \in \Sigma_\varphi$.

(*ii*) For every $A, B \in \Sigma_\varphi$ and every C of appropriate size, $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma_\varphi$. A set of matrices satisfying (*i*) and (*ii*) is called *upper multiplicative*.

(*iii*) For every $A, B \in \Sigma_\varphi$ and every C of appropriate size, $\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in \Sigma_\varphi$. A set of matrices satisfying (*i*) and (*iii*) is called *lower multiplicative*.

Therefore, [Coh06, Theorem 7.1.2 & Proposition 7.1.3] apply here to tell us that $R^\varphi(S)$ is actually a subring of S over which every matrix can be obtained as a solution of a matrix equation over $\varphi(R)$.

Theorem 3.3.6. *Let $\varphi : R \to S$ be a ring homomorphism and let Σ_φ be the set of all matrices over R whose image is invertible over S. Then $R^\varphi(S)$ is a subring of S containing $\varphi(R)$ and, for every $n \times m$ matrix M over $R^\varphi(S)$, there exists*

1. *an integer $r \geq 0$,*
2. *a matrix B over R of the form $B = \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}$ with $r + n$ rows and blocks B_i with m, r and n columns, respectively, with $\begin{pmatrix} B_2 & B_3 \end{pmatrix} \in \Sigma_\varphi$,*
3. *and a matrix u over S of the form $u = \begin{pmatrix} I_m \\ U \\ M \end{pmatrix}$, where U has size $r \times m$,*

such that if $A = \varphi(B) = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$, with $A_i = \varphi(B_i)$, we have $Au = 0$.

Remark 3.3.7. In fact, the matrix u in Theorem 3.3.6 can be directly taken over $R^{\varphi}(S)$. On the one hand, once we know that $R^{\varphi}(S)$ is a subring of S containing $\varphi(R)$, we can consider φ as a homomorphism $\varphi : R \to R^{\varphi}(S)$. On the other hand, the set Σ'_{φ} of matrices over R whose images under φ become invertible over $R^{\varphi}(S)$ coincides with Σ_{φ}. Indeed, we always have $\Sigma'_{\varphi} \subseteq \Sigma_{\varphi}$, and conversely, given $A \in \Sigma_{\varphi}$, $R^{\varphi}(S)$ contains by definition all the entries of $\varphi(A)^{-1}$, so that $\varphi(A)^{-1}$ is a matrix over $R^{\varphi}(S)$. This shows that $\Sigma_{\varphi} \subseteq \Sigma'_{\varphi}$. Therefore, $R^{\varphi}(S) = R^{\varphi}(R^{\varphi}(S))$, and applying [Coh06, Proposition 7.1.3] directly to $\varphi : R \to R^{\varphi}(S)$, we obtain the desired result.

Using this, we shall see in the next proposition that for an element x in $R^{\varphi}(S)$, there exist $r \geq 0$, invertible matrices P, Q over $R^{\varphi}(S)$ (in particular over S) and a matrix X over $\varphi(R)$ such that $I_r \oplus x = PXQ$. Hence, if x is invertible over S, then X is invertible over S and hence over $R^{\varphi}(S)$ because all the entries in X^{-1} belong to $R^{\varphi}(S)$ by definition. Thus, x^{-1} is an entry of the matrix $(PXQ)^{-1}$ over $R^{\varphi}(S)$, and hence $x^{-1} \in R^{\varphi}(S)$. In other words, the rational closure is division closed in S and since it contains $\varphi(R)$, we deduce that $\mathcal{D}_{\varphi(R),S} \subseteq R^{\varphi}(S)$. □

As a consequence of this theorem we obtain the following important result, which is a weak form of *Cramer's rule* (cf. [Coh06, Proposition 7.1.5]).

Proposition 3.3.8. *Let $\varphi : R \to S$ be a ring homomorphism. For every matrix M over $R^{\varphi}(S)$, there exist an integer $r \geq 0$, a matrix M' over $\varphi(R)$, and invertible matrices P and Q over $R^{\varphi}(S)$ such that*

$$\begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} = PM'Q$$

Proof. Let M be an $n \times m$ matrix over $R^{\varphi}(S)$, and let Σ_{φ}, A and u be as in Theorem 3.3.6, with u a matrix over $R^{\varphi}(S)$ as explained in Remark 3.3.7. We can express the equality $Au = 0$ in the form

$$A_1 + \begin{pmatrix} A_2 & A_3 \end{pmatrix} \begin{pmatrix} U \\ M \end{pmatrix} = 0,$$

and hence we have

$$\begin{pmatrix} A_2 & -A_1 \end{pmatrix} = \begin{pmatrix} A_2 & A_3 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} I_r & U \\ 0 & I_m \end{pmatrix}$$

Now, $M' = \begin{pmatrix} A_2 & -A_1 \end{pmatrix}$ is a matrix over $\varphi(R)$, $\begin{pmatrix} I_r & U \\ 0 & I_m \end{pmatrix}$ is an invertible matrix over $R^{\varphi}(S)$ with inverse Q and, since $\begin{pmatrix} A_2 & A_3 \end{pmatrix}$ is the image of an element of Σ_{φ}, it is invertible over S and its inverse P is a matrix over $R^{\varphi}(S)$. Thus,

$$\begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} = PM'Q$$

□

In particular, the previous result applies to universal localizations because of the next remark that we keep as a lemma.

Lemma 3.3.9. *Let R be a ring, Σ a set of matrices over R and R_Σ the universal localization of R at Σ, with natural map $\varphi : R \to R_\Sigma$. Then $R^\varphi(R_\Sigma) = R_\Sigma$.*

Proof. On the one hand, $R^\varphi(R_\Sigma)$ is a subring of R_Σ containing $\varphi(R)$. On the other hand, if $A \in \Sigma$, then by definition $\varphi(A)$ is invertible over R_Σ and hence, again by definition, the entries of $\varphi(A)^{-1}$ lie in $R^\varphi(R_\Sigma)$. But these entries, together with the elements of $\varphi(R)$, are the generators of R_Σ (see Remark 3.1.11), from where $R^\varphi(R_\Sigma) = R_\Sigma$. □

Recall that, since the universal Σ-inverting map $\varphi : R \to R_\Sigma$ is epic, H. Li's result [Li20, Theorem 8.1], which we stated in Proposition 2.1.9, tells us that the induced map $\varphi^\sharp : \mathbb{P}(R_\Sigma) \to \mathbb{P}(R)$ is injective.

We already discussed after Properties 1.2.2 that a ring with a Sylvester matrix rank function must have IBN, so in particular, if R_Σ is non-zero and $\mathbb{P}(R_\Sigma)$ is non-empty, it must be the case that the set Σ consists only of square matrices. Moreover, since every $n \times n$ matrix $A \in \Sigma$ is to become invertible over R_Σ, every Sylvester matrix rank function rk on R that is in the image of $\varphi^\sharp$ must verify $\mathrm{rk}(A) = n$.

One can wonder whether this necessary condition is also sufficient, and in this sense Proposition 3.3.8 gives us the only way the rank could be extended. If we had $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}')$ for some $\mathrm{rk}' \in \mathbb{P}(R_\Sigma)$ and a matrix M over R_Σ, then from the relation

$$\begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} = PM'Q,$$

where P and Q are invertible and $M' = \varphi(M'')$, we should have

$$\mathrm{rk}'(M) = \mathrm{rk}' \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} - r = \mathrm{rk}'(PM'Q) - r = \mathrm{rk}'(\varphi(M'')) - r = \mathrm{rk}(M'') - r.$$

Of course, if we have $\mathrm{rk} \in \mathbb{P}(R)$ and we construct rk' by setting $\mathrm{rk}'(M) = \mathrm{rk}(M'') - r$, one has to check that rk' is well-defined, since for instance the expression in Proposition 3.3.8 may not be unique, and then prove that it defines a Sylvester matrix rank function on R_Σ.

It turns out that this can be done through Malcolmson's criterion (cf. [Sch85, Theorem 4.2]), that characterizes when two matrices in R_Σ are equal. Thus we have the following result due to Schofield (cf. [Sch85, Theorem 7.4] or [Li20, Theorem 8.4] for a more recent approach). Since they are written in the language of Sylvester map rank functions, observe that if $A \in \Sigma$ is $n \times n$ and we consider the maps $r_A : R^n \to R^n$ and $id_{R^n} = r_{I_n}$, the relation between these versions of rank functions ([Sch85, Lemma 7.2]) gives us that $\mathrm{rk}(A) = \mathrm{rk}(r_A)$, $\mathrm{rk}(\mathrm{id}_{R^n}) = \mathrm{rk}(I_n) = n$.

Proposition 3.3.10. *Let R be a ring, Σ a set of square matrices over R and let* rk *be a Sylvester matrix rank function on R such that $\mathrm{rk}(A) = n$ for every $n \times n$ matrix $A \in \Sigma$. Then R_Σ is non-zero and* rk *extends to a Sylvester matrix rank function on R_Σ. If $\varphi : R \to R_\Sigma$ is the natural map, then*

$$\mathrm{im}\,\varphi^\sharp = \{\mathrm{rk} \in \mathbb{P}(R) : \mathrm{rk}(A) = n \text{ for every } n \times n\ A \in \Sigma\}.$$

We finish the section with a last comment regarding closures. At some point we shall be working with subrings R not of division rings but of regular rings $\mathcal{U}$. We would like to define some sort of *regular closure*, defined as the smallest regular subring of $\mathcal{U}$ containing R, and whose construction is related to R in a way similar to that of Proposition 3.3.2. Nevertheless, the main obstacle is that the intersection of regular subrings of $\mathcal{U}$ may not be regular (see [Goo91, Example 1.10]).

What allows us to guarantee the validity of this procedure for division rings is the uniqueness of the inverse of any element x, which must therefore lie in every division subring containing x and hence in their intersection. We show in Chapter 4 that, under the assumption that $\mathcal{U}$ is $*$-regular, for every $x \in \mathcal{U}$ there exists a particular "pseudo-inverse" y (i.e. such that $xyx = x$) that must lie inside every $*$-regular subring, hence providing the notion of *$*$-regular closure* of a $*$-subring of $\mathcal{U}$ ([AG17, Proposition 6.2]).

3.4 Crossed products, locally indicable groups and Hughes-free division rings of fractions

In Section 3.1 we introduced one particular instance of division ring of fractions, namely, the universal one. In this section we introduce another specific example of division ring of fractions which is associated to crossed products $E * G$ of a division ring E with a locally indicable group G. This division ring of fractions is called Hughes-free after I.Hughes, who introduced them in [Hug70] and proved that, if one exists for the crossed product $E * G$, then it is unique up to $E * G$-isomorphism. Prior to introducing here its defining property, we dedicate a subsection to crossed products and introduce locally indicable groups.

3.4.1 Crossed products

In this subsection we introduce in details the basic properties and behavior of crossed products (sometimes referred to as crossed product group rings). The natural example of a ring that may be constructed from a ring R and a group G is the *group ring* $R[G]$, a ring that is free as a (left) R-module with basis G, i.e., in which every element can be uniquely written in the form

$$\sum_{g\in G} r_g g$$

with only a finite number of non-zero $r_g \in R$, and in which addition is natural and multiplication is extended linearly from the multiplication in R and G. Thus, $gr = rg$ for all $g \in G$ and $r \in R$, and

$$\left(\sum_{g_1\in G} r_{g_1} g_1\right)\left(\sum_{g_2\in G} r_{g_2} g_2\right) = \sum_{g\in G}\left(\sum_{g_1 g_2 = g} r_{g_1} r_{g_2}\right) g.$$

If e denotes the neutral element in G, then $1_{R[G]} = 1e$, and we can embed R into $R[G]$ through the map $R \to R[G]$ sending $r \mapsto re$. Moreover, the map $G \to R[G]$ sending

$g \mapsto g$ is an injective group homomorphism, so that G can be seen as a subgroup of the group of units of $R[G]$, denoted $R[G]^\times$.

The crossed products $R * G$ generalize this construction by allowing modifications in the way in which the representatives of the elements of the group G are multiplied with each other and with the elements of R, and still retain most of the previous properties for group rings.

Nevertheless, this notion may be misleading when reading about it for the first time: different references may introduce crossed products from different perspectives, causing the elements that are considered part of the structure or their defining properties to differ slightly. For instance, they can be introduced as rings with a particular module structure relative to R and G, or as free R-modules in a fixed basis (which is a copy of G) that happens to become a ring under certain multiplication rules. From the former point of view we can change the basis and multiplication rules without changing the ring, while in the latter the basis is part of the structure, turning "equality" into "isomorphism" when changing it. Sometimes, some extra conditions are assumed on the rules defining multiplication in order to have a natural embedding of R. Because of this, we try to give a detailed treatment starting from the first perspective, which corresponds to [Pas89].

Definition 3.4.1. Let R be a ring and let G be a group. A *crossed product* $R * G$ of R and G is a ring that contains R, which is free as a left R-module with R-basis a copy $\{u_g : g \in G\}$ of G, and in which addition is the natural one and the ring multiplication is determined by the following two rules:

- There is a map of sets $\alpha : G \times G \to R^\times$, where $R^\times$ denotes the group of units of R, called the *twisting*, such that $u_g \cdot u_h = \alpha(g,h) \cdot u_{gh}$ for every $g, h \in G$.
- There is a map of sets $\sigma : G \to \operatorname{Aut}(R)$, called the *action*, such that $u_g \cdot r = \sigma(g)(r) \cdot u_g$ for every $r \in R$ and $g \in G$. We usually denote $\sigma(g)$ by σ_g to ease the notation.

In particular, every element $x \in R*G$ is uniquely written as $\sum_{g\in G} r_g u_g$ with only finitely many non-zero r_g. The finite set $\operatorname{supp}(x) = \{g \in G : r_g \neq 0\}$ is called the *support of x*.

In the first place, notice that $R * G$ is a ring that happens to have a relation with R and G, and in which multiplication can be defined through the twisting and the action. When we are given a crossed product this is enough information to start working, but if we want to construct a crossed product, we have to take into account that having such maps is not sufficient: the ring structure imposes several conditions on them (cf. [Pas89, Lemma 1.1] or [Sán08, Lemma 4.2]).

Lemma 3.4.2. *The associativity of $R * G$ is equivalent to the following two conditions on the action and twisting for all $g, h, k \in G$:*

(*i*) $\alpha(g,h)\alpha(gh,k) = \sigma_g(\alpha(h,k))\alpha(g,hk)$.

(*ii*) $\mu_{g,h}\sigma_{gh} = \sigma_g\sigma_h$, *where* $\mu_{g,h}(r) = \alpha(g,h)r\alpha(g,h)^{-1}$ *for every* $r \in R$, *an automorphism of R.*

Proof. Since the multiplication is extended linearly from the multiplication of "monomials" ru_g, the associativity of $R * G$ is equivalent to the following equality for every $g, h, k \in G$, $r, s, t \in R$,

$$[(ru_g)(su_h)](tu_k) = (ru_g)[(su_h)(tu_k)].$$

Developing the left-hand-side expression,

$$\begin{aligned}
[(ru_g)(su_h)](tu_k) &= (r\sigma_g(s)\alpha(g,h)u_{gh})(tu_k) \\
&= r\sigma_g(s)\alpha(g,h)\sigma_{gh}(t)\alpha(gh,k)u_{ghk} = \\
&= r\sigma_g(s)\alpha(g,h)\sigma_{gh}(t)\alpha(g,h)^{-1}\alpha(g,h)\alpha(gh,k)u_{ghk} \\
&= r\sigma_g(s)\mu_{g,h}(\sigma_{gh}(t))\alpha(g,h)\alpha(gh,k)u_{ghk},
\end{aligned}$$

and developing the right-hand-side expression,

$$\begin{aligned}
(ru_g)[(su_h)(tu_k)] &= (ru_g)(s\sigma_h(t)\alpha(h,k)u_{hk}) \\
&= r\sigma_g(s)\sigma_g(\sigma_h(t))\sigma_g(\alpha(h,k))\alpha(g,hk)u_{ghk}.
\end{aligned}$$

If (i) and (ii) hold, then we can see that these expressions coincide and give the associativity. Conversely, if these expressions are equal, and since σ_g is an automorphism, we have for $r = s = t = 1$ that

$$\alpha(g,h)\alpha(gh,k) = \sigma_g(\alpha(h,k))\alpha(g,hk),$$

and then for $r = s = 1$ that for every $t \in R$,

$$\mu_{g,h}(\sigma_{gh}(t))\alpha(g,h)\alpha(gh,k) = \sigma_g(\sigma_h(t))\sigma_g(\alpha(h,k))\alpha(g,hk),$$

from where equality (and invertibility) in the previous expression implies $\mu_{g,h}(\sigma_{gh}(t)) = \sigma_g(\sigma_h(t))$, what gives (ii). □

From here, it is possible to extract relations that allow us to identify 1_{R*G} and to show that each element u_g in the basis is invertible in $R * G$. In the following, let e denote the neutral element of G.

Properties 3.4.3. *Let $R * G$ be a crossed product with twisting α and action σ. The following hold.*

(1) *For every $g, k \in G$, $\alpha(g,e) = \sigma_g(\alpha(e,k))$.*

(2) *For every $g \in G$, $\alpha(e,g) = \alpha(e,e)$.*

(3) $\sigma_e = \mu_{e,e}$. *In particular $\sigma_e(\alpha(e,e)) = \alpha(e,e)$.*

(4) *For every $g \in G$, $\sigma_g(\alpha(g^{-1},g)) = \alpha(g,g^{-1})\alpha(e,g)\alpha(g,e)^{-1}$.*

(5) *For every $g \in G$, $(\mu_{g^{-1},g})^{-1}\sigma_{g^{-1}} = \mu_{e,e}\sigma_g^{-1}$.*

Proof. Taking $h = e$ in the Lemma 3.4.2(i) we obtain that

$$\alpha(g,e)\alpha(g,k) = \sigma_g(\alpha(e,k))\alpha(g,k),$$

and since $\alpha(g,k)$ is a unit, we obtain (1). Furthermore, we obtain from (1) that $\sigma_g(\alpha(e,k)) = \sigma_g(\alpha(e,e))$ for every $k \in G$, what gives us (2) because σ_g is an automorphism. Similary, substituting $h = g^{-1}$ and $k = g$, we get

$$\alpha(g,g^{-1})\alpha(e,g) = \sigma_g(\alpha(g^{-1},g))\alpha(g,e),$$

from where (4) follows.

Now, substituting $g = h = e$ in Lemma 3.4.2(ii), we obtain that $\mu_{e,e}\sigma_e = \sigma_e\sigma_e$, and since σ_e is an automorphism, $\sigma_e = \mu_{e,e}$, what gives (3). Using this and taking $g = h^{-1}$ in the same expression, we get

$$\sigma_{h^{-1}}\sigma_h = \mu_{h^{-1},h}\sigma_e = \mu_{h^{-1},h}\mu_{e,e},$$

from where (5) is deduced. □

The previous properties are just needed to give a quick overview to the structure of $R * G$. Compare the properties with the ones on group rings.

Corollary 3.4.4. *Let $R * G$ be a crossed product in the basis $\{u_g : g \in G\}$, with twisting α and action σ. The following hold.*

- (i) $1_{R*G} = \alpha(e,e)^{-1}u_e$. *For $r \in R$, we denote the element $r\alpha(e,e)^{-1}u_e$ by $r1_{R*G}$, and we have that $r1_{R*G} \cdot u_g = ru_g = u_g \cdot \sigma_g^{-1}(r)1_{R*G}$ for every $g \in G$.*
- (ii) *The embedding of R into $R * G$ is given by $r \to r1_{R*G}$.*
- (iii) *For every $g \in G$ and for every unit $r \in R^\times$, the element ru_g is invertible in $R * G$, and $(ru_g)^{-1} \in R^\times u_{g^{-1}}$.*
- (iv) *The set $R^\times G = \{ru_g : r \in R, g \in G\}$ is a subgroup of the group of units of $R * G$. Moreover, $R^\times = R^\times u_e$ is a normal subgroup of $R^\times G$ and $R^\times G/R^\times \cong G$ as groups.*

Proof.
(i) Observe that it suffices to check the property over monomials ru_g. Using Properties 3.4.3(1) with $k = e$, and for every $s \in R$,

$$\begin{aligned}(ru_g)(s\alpha(e,e)^{-1}u_e) &= r\sigma_g(s)\sigma_g(\alpha(e,e)^{-1})\alpha(g,e)u_g \\ &\overset{(1)}{=} r\sigma_g(s)\sigma_g(\alpha(e,e)^{-1})\sigma_g(\alpha(e,e))u_g = r\sigma_g(s)u_g\end{aligned}$$

while using Properties 3.4.3(2) and (3),

$$\begin{aligned}(s\alpha(e,e)^{-1}u_e)(ru_g) &= s\alpha(e,e)^{-1}\sigma_e(r)\alpha(e,g)u_g \\ &\overset{(2),(3)}{=} s\alpha(e,e)^{-1}\alpha(e,e)r\alpha(e,e)^{-1}\alpha(e,e)u_g = sru_g.\end{aligned}$$

Taking $s = 1$, we see that $1_{R*G} = \alpha(e,e)^{-1}u_e$. Now, for any $t \in R$, the first equality with $r = 1$ and $s = \sigma_g^{-1}(t)$ shows that $u_g \cdot \sigma_g^{-1}(t)1_{R*G} = tu_g$, while the second one for $r = 1$ and $s = t$ shows that $(t1_{R*G}) \cdot u_g = tu_g$.

(ii) The map is well-defined and injective, since $\{u_g\}$ is a basis and hence $r1_{R*G} = 0$ if and only if $r\alpha(e,e)^{-1} = 0$, if and only if $r = 0$. It is additive because of the way the addition is defined in $R * G$, and it is a ring homomorphism because, using Properties 3.4.3(2) and (3),

$$\begin{aligned}(r1_{R*G})(s1_{R*G}) &= r\alpha(e,e)^{-1}\sigma_e(s)\sigma_e(\alpha(e,e))^{-1}\alpha(e,e)u_e \\ &\overset{(2),(3)}{=} rs\alpha(e,e)^{-1}u_e = rs1_{R*G}.\end{aligned}$$

(iii) For $r \in R^\times$ and $g \in G$, consider $x = \sigma_g^{-1}(r^{-1})\alpha(e,e)^{-1}\alpha(g^{-1},g)^{-1}u_{g^{-1}}$. On the one hand, we have from Properties 3.4.3(5) that

$$\begin{aligned}x(ru_g) &= \sigma_g^{-1}(r^{-1})\alpha(e,e)^{-1}\alpha(g^{-1},g)^{-1}\sigma_{g^{-1}}(r)\alpha(g^{-1},g)u_e \\ &= \sigma_g^{-1}(r^{-1})\alpha(e,e)^{-1}(\mu_{g^{-1},g})^{-1}[(\sigma_{g^{-1}}(r)]u_e \\ &\overset{(5)}{=} \sigma_g^{-1}(r^{-1})\alpha(e,e)^{-1}\mu_{e,e}(\sigma_g^{-1}(r))u_e \\ &= \sigma_g^{-1}(r^{-1})\sigma_g^{-1}(r)1_{R*G} = 1_{R*G}.\end{aligned}$$

while on the other hand, using Properties 3.4.3(1),(2) and (4),

$$\begin{aligned}(ru_g)x &= r\sigma_g(\sigma_g^{-1}(r^{-1}))\sigma_g(\alpha(e,e))^{-1}\sigma_g(\alpha(g^{-1},g))^{-1}\alpha(g,g^{-1})u_e \\ &\overset{(1),(4)}{=} \alpha(g,e)^{-1}\alpha(g,e)\alpha(e,g)^{-1}\alpha(g,g^{-1})^{-1}\alpha(g,g^{-1})u_e \\ &= \alpha(e,g)^{-1}u_e \overset{(2)}{=} 1_{R*G}.\end{aligned}$$

Thus, $x = (ru_g)^{-1}$, and this finishes the proof of *(iii)*.

(iv) From *(i)* and *(iii)*, $1_{R*G} \in R^\times G$ and every element in $R^\times G$ is invertible with inverse in $R^\times G$. Since the product of elements of the form ru_g with r a unit is again of this form, $R^\times G$ is a subgroup of the group of units of $R * G$. Now, observe that $R^\times := \{r1_{R*G} : r \in R^\times\} = \{ru_e : r \in R^\times\} = R^\times u_e$. Since in *(iii)* we noticed that $(ru_g)^{-1} \in R^\times u_{g^{-1}}$, we have for every $r, s \in R^\times$,

$$(ru_e)(su_e)^{-1} \in R^\times u_e \quad \text{and} \quad (ru_g)(su_e)(ru_g)^{-1} \in R^\times u_e.$$

Therefore, $R^\times$ is a normal subgroup of $R^\times G$. Consider the natural map

$$\phi : R^\times G/R^\times \to G$$

given by $R^\times ru_g \mapsto g$. The map ϕ is well-defined, because if $R^\times ru_g = R^\times su_h$, then $(ru_g)(su_h)^{-1} \in R^\times u_e \cap R^\times u_{gh^{-1}}$, and since $\{u_g : g \in G\}$ is a basis of $R * G$, it must be the case that $gh^{-1} = e$, i.e., $g = h$. Moreover, we can see from the definition that it is bijective, and it is a group homomorphism because for every $g, h \in G$, r, s units in R, $(ru_g)(su_h) \in R^\times u_{gh}$. □

Therefore, we can see from this corollary the resemblance in structure between group rings and crossed products. In fact, a group ring is a particular case of crossed product, in which the twisting α and the action σ are trivial (i.e. $\alpha(g,h) = 1_R$ and $\sigma_g = \mathrm{id}_R$ for every $g, h \in G$). One of the properties that does not hold for general crossed products is that the group G can be embedded as a subgroup of units of $R * G$. Nevertheless, the representatives $\{u_g : g \in G\}$ of elements of G are still invertible, and we can recover G as the quotient $R^\times G / R^\times$.

As we mentioned earlier, constructing a crossed product from scratch is not as easy as just giving two maps telling us how to multiply elements of the basis with each other and with the elements of R. We need those maps to satisfy Lemma 3.4.2, and this may not be automatic to verify. However, there are two situations in which we can construct a crossed product from another crossed product, one involving subgroups of G and one involving particular R-rings (S, φ).

Definition 3.4.5. Given a crossed product $R * G$ in the basis $\{u_g : g \in G\}$ and a subgroup $H \leq G$, we define

$$R * H = \{a \in R * G : \mathrm{supp}(a) \subseteq H\},$$

which is a crossed product in the basis $\{u_h : h \in H\}$, in which the action and twisting are just the restrictions of the ones in $R * G$.

Observe that $1_{R*G} \in R*H$ and $R*H$ is closed under substraction and multiplication, so it is indeed a subring of $R * G$. In this case, the restrictions of α and σ automatically satisfy Lemma 3.4.2 because α and σ do (or equivalently because $R * G$ is associative).

The second instance of crossed product that can be constructed from $R * G$ is given in the following proposition.

Proposition 3.4.6. *Let $\varphi : R \to S$ be a ring homomorphism and assume that for every automorphism τ of R, there exists a unique automorphism $\tilde{\tau}$ of S such that the following diagram commutes*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\tau"'] & S \arrow[d, dotted, "\tilde{\tau}"] \\ R \arrow[r, "\varphi"] & S. \end{tikzcd}$$

Then, every crossed product structure $R * G$ defines a crossed product structure $S * G$ and a ring homomorphism $\tilde{\varphi} : R * G \to S * G$ such that $\tilde{\varphi}(r 1_{R*G}) = \varphi(r) 1_{S*G}$, i.e., such that the following commutes

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook] & S \arrow[d, hook] \\ R * G \arrow[r, "\tilde{\varphi}"] & S * G. \end{tikzcd}$$

Proof. Assume that we are given a crossed product $R * G$ of R and G in the basis $\{u_g : g \in G\}$ with twisting α and action σ. Define $\tilde{\alpha} : G \times G \to S^\times$ to be the composition

$\tilde{\alpha} = \varphi \circ \alpha$, and $\tilde{\sigma} : G \to \mathrm{Aut}(S)$ given by $g \to \tilde{\sigma}_g$, where $\tilde{\sigma}_g$ is the unique automorphism of S with $\tilde{\sigma}_g \circ \varphi = \varphi \circ \sigma_g$. Let us check that $\tilde{\tau}$ and $\tilde{\sigma}$ satisfy the conditions on Lemma 3.4.2 (i): From the definitions of $\tilde{\alpha}$ and $\tilde{\sigma}$, and since φ is a ring homomorphism, we have that for every $g, h, k \in G$,

$$\begin{aligned}\tilde{\alpha}(g,h)\tilde{\alpha}(gh,k) &= \varphi[\alpha(g,h)\alpha(gh,k)] = \varphi[\sigma_g(\alpha(h,k))\alpha(g,hk)] \\ &= \tilde{\sigma}_g[\varphi(\alpha(h,k))]\varphi[\alpha(g,hk)] = \tilde{\sigma}_g(\tilde{\alpha}(h,k))\tilde{\alpha}(g,hk).\end{aligned}$$

(ii): For every $g, h \in G$, let $\tilde{\mu}_{g,h}$ be the automorphism of S given by conjugation by $\tilde{\alpha}(g,h)$. On the one hand, we have that

$$\tilde{\sigma}_g\tilde{\sigma}_h\varphi = \tilde{\sigma}_g\varphi\sigma_h = \varphi\sigma_g\sigma_h.$$

On the other hand, for every $r \in R$,

$$\begin{aligned}\tilde{\mu}_{g,h}\tilde{\sigma}_{gh}\varphi(r) &= \tilde{\mu}_{g,h}(\varphi\sigma_{gh}(r)) = \tilde{\alpha}(g,h)\varphi\sigma_{gh}(r)\tilde{\alpha}(g,h)^{-1} \\ &= \varphi[\alpha(g,h)\sigma_{gh}(r)\alpha(g,h)^{-1}] = \varphi[\mu_{g,h}\sigma_{gh}(r)] \\ &= \varphi\sigma_g\sigma_h(r),\end{aligned}$$

from where $\tilde{\mu}_{g,h}\tilde{\sigma}_{gh}\varphi = \varphi\sigma_g\sigma_h$. This implies that $\tilde{\sigma}_g\tilde{\sigma}_h$ and $\tilde{\mu}_{g,h}\tilde{\sigma}_{gh}$ are two automorphisms of S for which the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\sigma_g\sigma_h"'] & S \arrow[d, "\tilde{\sigma}_g\tilde{\sigma}_h"', "\tilde{\mu}_{g,h}\tilde{\sigma}_{gh}"] \\ R \arrow[r, "\varphi"] & S. \end{tikzcd}$$

Since $\sigma_g\sigma_h$ is an automorphism of R, the uniqueness in the hypothesis implies that $\tilde{\sigma}_g\tilde{\sigma}_h = \tilde{\mu}_{g,h}\tilde{\sigma}_{gh}$.

If we construct the free left S-module with basis $\{u_g : g \in G\}$ and we define a multiplication by setting $u_gu_h = \tilde{\alpha}(g,h)u_{gh}$ and $u_gs = \tilde{\sigma}_g(s)u_g$, and extending this linearly, we obtain from the previous properties an associative ring S' with $1_{S'} = \tilde{\alpha}(e,e)^{-1}u_e$ in which S embeds through $s \mapsto s1_{S'}$ (as in the proof of Corollary 3.4.4). Thus, $S' = S * G$.

Finally, define the map $\tilde{\varphi} : R * G \to S * G$ with $\tilde{\varphi}\left(\sum r_gu_g\right) = \sum \varphi(r_g)u_g$, which is additive because φ is a ring homomorphism, and sends

$$\tilde{\varphi}(1_{R*G}) = \varphi(\alpha(e,e)^{-1})u_e = \tilde{\alpha}(e,e)^{-1}u_e = 1_{S*G}.$$

Moreover, it is a ring homomorphism, since

$$\begin{aligned}\tilde{\varphi}((ru_g)(r'u_h)) &= \tilde{\varphi}(r\sigma_g(r')\alpha(g,h)u_{gh}) = \varphi(r)\varphi(\sigma_g(r'))\varphi(\alpha(g,h))u_{gh} \\ &= \varphi(r)\tilde{\sigma}_g(\varphi(r'))\tilde{\alpha}(g,h)u_{gh} = (\varphi(r)u_g)(\varphi(r')u_h) \\ &= \tilde{\varphi}(ru_g)\tilde{\varphi}(r'u_h)\end{aligned}$$

Finally, $\tilde{\varphi}(r1_{R*G}) = \varphi(r)\varphi(\alpha(e,e))^{-1}u_e = \varphi(r)1_{S*G}$. □

In a situation like the one in Proposition 3.4.6, it will also be interesting for Section 3.5 to understand the $R * G$-module structure of $S * G$. For this purpose, assume that R is a subring of S and that the crossed product $R * G$ can be *extended* to a crossed product $S * G$. By this we mean that the twisting α on $R * G$ and $S * G$ coincide, the action $\tilde{\sigma}$ on $S * G$ is such that for every $g \in G$, $\tilde{\sigma}_{g_{|R}} = \sigma_g$, and hence the map $R * G \to S * G$ sending $ru_g \mapsto su_g$ is an injective ring homomorphism.

Lemma 3.4.7. *Let R be a subring of a ring S and fix any crossed product $R*G$. If $R*G$ extends to a crossed product $S*G$, then the left $R*G$-modules $S*G$ and $(R*G)\otimes_R S$ are isomorphic. Similarly, the right $R*G$-modules $S*G$ and $S \otimes_R (R*G)$ are isomorphic.*

Proof. Let us prove the statement for the left $R * G$-modules. Define the map

$$\begin{array}{cccc} \varphi : & (R*G) \times S & \to & S*G \\ & \left(\sum_{g\in G} r_g u_g, s\right) & \mapsto & \sum_{g\in G} r_g \tilde{\sigma}_g(s) u_g \end{array}$$

This map is R-biadditive, since it is bilinear and, using Corollary 3.4.4(i) and (ii), we can see that, for every $r \in R$,

$$\begin{aligned} \varphi\left(\left[\sum_{g\in G} r_g u_g\right] \cdot r1_{R*G}, s\right) &= \varphi\left(\sum_{g\in G} r_g \sigma_g(r) u_g, s\right) = \sum_{g\in G} r_g \sigma_g(r) \tilde{\sigma}_g(s) u_g \\ &= \sum_{g\in G} r_g \tilde{\sigma}_g(rs) u_g = \varphi\left(\sum_{g\in G} r_g u_g, rs\right) \end{aligned}$$

Therefore, φ induces a well-defined homomorphism of abelian groups, say,

$$\psi : (R*G) \otimes_R S \to S*G,$$

that sends $\left(\sum_{g\in G} r_g u_g\right) \otimes s \mapsto \sum_{g\in G} r_g \tilde{\sigma}_g(s) u_g$. If we now take two elements $x = \sum_{h\in G} r'_h u_h$ and $y = \sum_{k\in G} r_k u_k$ in $R*G$, one can check that

$$\begin{aligned} \psi(x \cdot (y \otimes s)) &= \sum_{g\in G} \left[\sum_{hk=g} r'_h \sigma_h(r_k) \alpha(h,k) \tilde{\sigma}_g(s)\right] u_g \\ x \cdot \psi(y \otimes s) &= \sum_{g\in G} \left[\sum_{hk=g} r'_h \tilde{\sigma}_h(r_k) \tilde{\sigma}_h(\tilde{\sigma}_k(s)) \alpha(h,k)\right] u_g \end{aligned}$$

These two elements are equal, since $\tilde{\sigma}_{h_{|R}} = \sigma_h$ and, from Lemma 3.4.2(ii),

$$\alpha(h,k)\tilde{\sigma}_g(s) = (\alpha(h,k)\tilde{\sigma}_g(s)\alpha(h,k)^{-1})\alpha(h,k) \overset{(ii)}{=} \tilde{\sigma}_h(\tilde{\sigma}_k(s))\alpha(h,k)$$

and hence, since ψ is additive, this suffices to claim that ψ is a homomorphism of left $R*G$-modules. In addition, it is surjective, since the element $\sum_{g\in G} s_g u_g \in S*G$ has $\sum_{g\in G}[u_g \otimes \tilde{\sigma}_g^{-1}(s_g)]$ as a preimage. Finally, to prove injectivity, it is useful to observe that a generic element $x \in (R*G)\otimes_R S$ can be written as

$$x = \sum_{i=1}^{n} \left[\left(\sum_{g\in G} r_g^{(i)} u_g\right) \otimes s_i\right] = \sum_{g\in G} u_g \otimes \left[\sum_{i=1}^{n} \sigma_g^{-1}(r_g^{(i)}) s_i\right]$$

Observe then that $\psi(x) = 0$ if and only if $\tilde{\sigma}_g(\sum_{i=1}^{n} \sigma_g^{-1}(r_g^{(i)})s_i) = 0$ for every $g \in G$. Since $\tilde{\sigma}_g$ is an automorphism, we deduce again from the previous expression that x must be zero. This finishes the proof.

For the right $R * G$-modules statement, one can proceed similarly to show that the map $\psi' : S \otimes_R (R * G) \to S * G$ sending $s \otimes \sum_{g\in G} r_g u_g \mapsto \sum_{g\in G} s r_g u_g$ is the desired isomorphism. $\square$

Up to now we have developed the basic theory of crossed products without further assumptions. However, the form of 1_{R*G} as $\alpha(e,e)^{-1}u_e$ and hence of the embedding $R \hookrightarrow R*G$ does not seem that natural. There are references that add an extra condition to the maps α and σ, namely, they assume that $\alpha(e,g) = \alpha(g,e) = 1_R$ for every $g \in G$, what implies in view of Properties 3.4.3(3) that $\sigma(e) = \mathrm{id}_R$ (cf. [Haz16, Subsection 1.1.4]). With these assumptions, 1_{R*G} coincides with the representative u_e of the neutral element $e \in G$, and the embedding of R is then just given by $r \to r u_e$. From our point of view, this can always be done via what Passman calls in [Pas89] a *diagonal change of basis*. By doing these, we also change the defining maps but not the ring, which is the base structure (see also [Sán08, Remarks 4.3]).

Lemma 3.4.8. *Let $R * G$ be a crossed product in the basis $\{u_g : g \in G\}$, with twisting α and action σ. If, for every $g \in G$, we choose a unit r_g, then the set $\{r_g u_g : g \in G\}$ is another R-basis of $R * G$ that still exhibits the basic crossed product structure.*

Proof. Since r_g is a unit, for every $g \in G$, it is clear that $\{r_g u_g : g \in G\}$ is another copy of G and an R-basis of $R * G$. Define $\tilde{\alpha} : G \times G \to R^\times$ by $\tilde{\alpha}(g,h) = r_g \sigma_g(r_h)\alpha(g,h) r_{gh}^{-1}$ and $\tilde{\sigma} : G \to \mathrm{Aut}(R)$ by $\tilde{\sigma}_g(r) = r_g \sigma_g(r) r_g^{-1}$, and observe that, if we denote $\tilde{u}_g = r_g u_g$, we have that

$$\tilde{u}_g \tilde{u}_h = r_g \sigma_g(r_h)\alpha(g,h) u_{gh} = \tilde{\alpha}(g,h)\tilde{u}_{gh}.$$
$$\tilde{u}_g r = r_g \sigma_g(r) u_g = \tilde{\sigma}_g(r)\tilde{u}_g.$$

Thus, after this relabeling we are seeing the same ring $R * G$ in the basis $\{\tilde{u}_g : g \in G\}$ with twisting $\tilde{\alpha}$ and action $\tilde{\sigma}$. $\square$

Remark 3.4.9. Take, in the previous lemma, $r_e = \alpha(e,e)^{-1}$. Then, for any choice of units r_g, $g \neq e$, we have by Properties 3.4.3(1), (2) and (3),

$$\tilde{\alpha}(g,e) = r_g \sigma_g(\alpha(e,e))^{-1}\alpha(g,e) r_g^{-1} \overset{(1)}{=} r_g r_g^{-1} = 1_R$$
$$\tilde{\alpha}(e,g) = \alpha(e,e)^{-1}\sigma_e(r_g)\alpha(e,g) r_g^{-1} \overset{(2),(3)}{=} r_g r_g^{-1} = 1_R,$$

and hence $\tilde{\sigma}_e(r) = \alpha(e,e)^{-1}\sigma_e(r)\alpha(e,e) = r$ for every $r \in R$, i.e., $\tilde{\sigma}_e = \mathrm{id}_R$. With this change, $\tilde{u}_e = \alpha(e,e)^{-1}u_e = 1_{R*G}$. Therefore, we can, and we will always assume that the identity element of $R * G$ is the representative of the neutral element $e \in G$. $\square$

For us, since the change of basis does not modify the ring (it can be just seen as a relabeling of its elements) nor the relations with R and G (it is the same free left R-module in another basis which is a copy of G), the outcome of a diagonal change of basis is the same object. This is why one can assume $\alpha(e,g) = \alpha(g,e) = 1_R$ without loss of generality. However, some authors consider the basis as part of the structure, what would make the outcome of this procedure a new crossed product isomorphic to the previous one.

We shall be mostly interested in crossed products in which the group G is the (multiplicative) group $\mathbb{Z}$. In this case we show that, up to a diagonal change of basis, any crossed product $R * \mathbb{Z}$ is a skew Laurent polynomial ring (see also [Sán08, Remark 4.6]).

Proposition 3.4.10. *Let R be any ring. Up to a diagonal change of basis, the crossed product $R*\mathbb{Z}$ can be seen as a skew Laurent polynomial ring $R[t^{\pm 1};\tau]$ for some automorphism τ of R.*

Proof. Let $\mathbb{Z}$ be given in multiplicative notation as $\mathbb{Z} = \{s^k : k \in \mathbb{Z}\}$. Observe first that if τ is an automorphism of R, then $R[t^{\pm 1};\tau]$ is a ring containing R which is free as an R-module in the basis $\{u_{s^i} := t^i : i \in \mathbb{Z}\}$, a copy of $\mathbb{Z}$. Define $\alpha : G \times G \to R^\times$ to be the trivial map $\alpha(s^k, s^l) = 1_R$ for every $k, l \in \mathbb{Z}$, and $\sigma : G \to \mathrm{Aut}(R)$ given by $\sigma_{t^k} = \tau^k$. Then $R[t^{\pm 1};\tau]$ has natural sum and multiplication determined by $t^k t^l = t^{k+l} = \alpha(s^k, s^l)t^{k+l}$ and $t^k r = \tau^k(r)t^k = \sigma_{s^k}(r)t^k$, i.e, $R[t^{\pm 1};\tau] = R * \mathbb{Z}$.

Conversely, let $R * \mathbb{Z}$ be a crossed product in the basis $\{u_{s^i} : i \in \mathbb{Z}\}$ with twisting α and action σ. Assume without loss of generality (see Remark 3.4.9) that $\alpha(1, s^k) = \alpha(s^k, 1) = 1_R$ for every $k \in \mathbb{Z}$, $\sigma_1 = \mathrm{id}_R$ and $1_{R*\mathbb{Z}} = u_1$. Now, for every non-zero $k \in \mathbb{Z}$, there exists a unit $r_k \in R$ such that $u_s^k = r_k u_{s^k}$, because we showed in Corollary 3.4.4 that $R^\times G$ is a group, any product $(ru_{s^k})(r'u_{s^l})$ lies in $R^\times u_{s^{k+l}}$ and the inverse of u_s lies in $R^\times u_{s^{-1}}$. Therefore, setting $\tilde{u}_1 = u_1 = 1_{R*\mathbb{Z}}$ and $\tilde{u}_{s^k} = u_s^k$ for any non-zero k, we can perform a diagonal change of basis from $\{u_{s^i} : i \in \mathbb{Z}\}$ to $\{\tilde{u}_{s^k} : k \in \mathbb{Z}\}$. For every $k, l \in \mathbb{Z}$ we have

$$\tilde{u}_{s^k}\tilde{u}_{s^l} = u_s^k u_s^l = u_s^{k+l} = \tilde{u}_{s^k s^l}$$

$$\tilde{u}_{s^k} r = u_s^k r = \sigma_s^k(r) u_s^k = \sigma_s^k(r)\tilde{u}_{s^k},$$

where we understand $u_s^0 = 1_{R*\mathbb{Z}} = u_1$ and $\sigma_s^0 = \mathrm{id}_R = \sigma_1$. Hence, with respect to the new basis, the twisting is trivial $\tilde{\alpha}(s^k, s^l) = 1_R$, and the action $\tilde{\sigma}$ is given by $\tilde{\sigma}_{s^i} = \sigma_s^k$. Thus, we have an R-ring isomorphism $R * \mathbb{Z} \cong R[t^{\pm 1};\sigma_s]$ where $u_s \mapsto t$. $\square$

Given a group G and a normal subgroup N of G, it is also important to understand better the relation between a crossed product $R * G$ and the subcrossed product $R * N$ defined on Definition 3.4.5. We shall frequently meet this situation when dealing with locally indicable groups. The following proposition corresponds to [Pas89, Lemma 1.3] (see also [Sán08, Lemma 4.7]).

Proposition 3.4.11. *Let $R*G$ be a crossed product of the ring R and the group G, and let N be a normal subgroup of G. Then*

$$R * G = (R * N) * G/N$$

Proof. Let $R*G$ be a crossed product in the basis $\{u_g : g \in G\}$ with twisting α and action σ, and assume, without loss of generality, that $u_e = 1_{R*G}$ (see Remark 3.4.9). Recall from Definition 3.4.5 that $R * N$ is the subcrossed product in the basis $\{u_n : n \in N\}$.

Let T be a transversal of N in G with $e \in T$, i.e., T contains exactly one representative in G for each conjugacy class in G/N, and we choose e as the representative of the trivial class. The set $\{u_t : t \in T\}$ is a copy of G/N, and we claim that

$$R * G = \bigoplus_{t\in T} (R * N)u_t.$$

On the one hand, if $g \in G$ and $t_g \in T$ is the unique element in T such that $Ng = Nt_g$, then there exists a unique $n \in N$ with $g = nt_g$, and therefore $u_g = (\alpha(n, t_g)^{-1}u_n)u_{t_g} \in (R*N)u_t$, from where $R*G = \sum_{t\in T}(R*N)u_t$. On the other hand, since T is a transversal, the equality $nt = n't'$ for some $n, n' \in N$ and $t, t' \in T$ only holds if $t = t'$ and consequently $n = n'$. Therefore, the expression $\sum_{t\in T}\left(\sum_{n\in N} r_{n,t}u_n\right) u_t = \sum_{t\in T}\sum_{n\in N} r_{n,t}\alpha(n, t)u_{nt}$, which covers each $g \in G$ only once, is zero if and only if $r_{n,t}\alpha(n, t) = 0$ for every $n \in N, t \in T$, or equivalently, if $r_{n,t} = 0$ for every $n \in N, t \in T$. Thus, the sum is direct and $\{u_t : t \in T\}$ is an $R * N$-basis of $R * G$.

Notice that for every $t \in T$, $n \in N$, and $r \in R$ we have

$$u_t(ru_n)u_t^{-1} \in Ru_{tnt^{-1}} \subseteq R * N \quad \text{and} \quad u_t^{-1}(ru_n)u_t \in Ru_{t^{-1}nt} \subseteq R * N$$

and therefore conjugation by u_t defines an automorphism μ_{u_t} of $R * N$. In addition, if $t_1, t_2 \in T$ and $t_{1,2} \in T$, $n_{1,2} \in N$ are the unique elements such that $t_1t_2 = n_{1,2}t_{1,2}$, then $u_{t_1}u_{t_2} = \alpha(t_1, t_2)\alpha(n_{1,2}, t_{1,2})^{-1}u_{n_{1,2}}u_{t_{1,2}}$.

Therefore, if we set $\tilde{u}_{Ng} := u_{t_g}$ where $Ng = Nt_g$, and we define

$$\tilde{\sigma} : G/N \to \mathrm{Aut}(R * N) \quad \text{and} \quad \tilde{\alpha} : G/N \times G/N \to (R * N)^\times$$

by $\tilde{\sigma}_{Ng} = \mu_{u_{t_g}}$ and $\tilde{\alpha}(Ng, Nh) = \alpha(t_g, t_h)\alpha(n_{g,h}, t_{g,h})^{-1}u_{n_{g,h}}$, respectively, where $n_{g,h} \in N$, $t_{g,h} \in T$ are the unique elements with $t_gt_h = n_{g,h}t_{g,h}$, then we can see from the previous expressions that $R * G = (R * N) * G/N$ in the basis $\{\tilde{u}_{Ng} : g \in G\}$, with twisting $\tilde{\alpha}$ and action $\tilde{\sigma}$. Moreover, the embedding of $R * N$ is the natural one $x \to x\tilde{u}_N = xu_e = x1_{R*G}$. □

We finish the introduction to crossed products by describing its relation to the notion of G-graded ring, following again the exposition in [Pas89].

Definition 3.4.12. Let G be a group. A ring S is G-*graded* if $S = \bigoplus_{g\in G} S_g$ where S_g is an additive subgroup for every $g \in G$, and $S_gS_h \subseteq S_{gh}$ for all $g, h \in G$. In this case, $R = S_e$ is a subring of S, called the *base ring* of S, and each S_g is an R-bimodule under left and right multiplication. A *strongly* G-*graded ring* is a G-graded ring S such that $S_gS_h = S_{gh}$ for all $g, h \in G$.

Remark. In analogy to the case of a crossed product, if S is a G-graded ring, then $1_S \in S_e$ and if $u \in S_g$ is invertible in S, then $u^{-1} \in S_{g^{-1}}$.

Indeed, by definition, 1_S admits a unique expression $1_S = \sum_{g\in G} s_g$ with $s_g \in S_g$ and only finitely many non-zero s_g. For every $g_0 \in G$ and every $r_{g_0} \in S_{g_0}$, the equality $r_{g_0} = r_{g_0}1_S = 1_Sr_{g_0}$, together with the property that $S_gS_h \subseteq S_{gh}$, implies in particular that $s_e r_{g_0} = r_{g_0} s_e = r_{g_0}$. Since this is true for every $g_0 \in G$, we deduce that for every $x \in S$, $xs_e = s_e x = x$, and by uniqueness of 1_S, we must have $1_S = s_e \in S_e$.

Similarly, if $u \in S_g$ is invertible in S and $u^{-1} = \sum_{g \in G} s_g$, then from the equality $uu^{-1} = u^{-1}u = 1_S \in S_e$ and the property $S_g S_h \subseteq S_{gh}$, we obtain in particular that $us_{g^{-1}} = s_{g^{-1}}u = 1_S$, and by uniqueness of the inverse, we must have $u^{-1} = s_{g^{-1}} \in S_{g^{-1}}$. □

Proposition 3.4.13. *Let R, S be rings and G a group. Then $S = R * G$ if and only if S is a G-graded ring with base ring $S_e = R$ such that, for every $g \in G$, S_g contains a unit u_g of S. Moreover, in any of these equivalent cases, S is actually strongly G-graded.*

Proof. If $S = R * G$ in the basis $\{u_g : g \in G\}$ and we set $S_g = Ru_g$, which is an additive subgroup, then by definition $S = \bigoplus_{g \in G} S_g$, u_g is a unit in S_g and we can see that $S_g S_h = S_{gh}$.

Conversely, let S be a G-graded ring and let u_g be an element of S_g invertible in S. From the previous remark, $u_g^{-1} \in S_{g^{-1}}$ and hence every $r \in S_g$ can be expressed as $(ru_g^{-1})u_g \in S_e u_g = Ru_g$. Therefore $S_g = Ru_g$. Similarly, $S_g S_h = S_{gh}$, and as a consequence $S = \bigoplus_{g \in G} Ru_g$ is a free left R-module with basis $\{u_g : g \in G\}$, a copy of G. Moreover, for every $g, h \in G$ and every $r \in R$, both $u_g r u_g^{-1}$ and $u_g u_h u_{gh}^{-1}$ belong to R, so we can define

$$\sigma : G \to \mathrm{Aut}(R) \quad \text{and} \quad \alpha : G \times G \to R^\times$$

by $\sigma_g = \mu_{u_g}$ and $\alpha(g,h) = u_g u_h u_{gh}^{-1}$, respectively, where $\mu_{u_g}(r) = u_g r u_g^{-1}$. From the definition, $u_g u_h = \alpha(g,h)u_{gh}$ and $u_g r = \sigma_g(r)u_g$ for every $g, h \in G$ and $r \in R$. Therefore, $S = R * G$. □

3.4.2 Locally indicable groups and Hughes-free division rings of fractions

In this subsection we introduce the Hughes-free division ring of fractions of a crossed product $E * G$ of a division ring E and a locally indicable group G. Let us first recall some of the basic properties and examples of locally indicable groups.

Definition 3.4.14. A group G is *indicable* if G is trivial or there exists a surjective group homomorphism $G \to \mathbb{Z}$. A group G is *locally indicable* if every finitely generated subgroup of G is indicable.

Remark. Equivalently, a group G is locally indicable if for every non-trivial finitely generated subgroup H of G, there exists a normal subgroup $N \lhd H$ such that H/N is infinite cyclic. □

Observe that, in particular, locally indicable groups are torsion-free, because for every $e \neq g \in G$ we can find a surjective homomorphism from the group generated by g to $\mathbb{Z}$, and hence g cannot have finite order. Let us list some of the examples of locally indicable groups that we shall meet later and some non-examples.

Example 3.4.15.

(1) Free groups are locally indicable, since every non-trivial finitely generated subgroup of a free group is again free by Nielsen-Schreier theorem, and hence admits a surjective homomorphism to $\mathbb{Z}$.

(2) Torsion-free abelian groups are locally indicable, since all its non-trivial finitely generated subgroups are again torsion-free abelian and hence, by the classification theorem of finitely generated abelian groups, isomorphic to $\mathbb{Z}^n$ for some $n \geq 1$.

(3) The family of locally indicable groups is closed under isomorphisms and under taking subgroups. However, it is not closed under homomorphic images: we have a natural surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}_2$, but $\mathbb{Z}_2$ is not locally indicable.

(4) Indicable groups need not be locally indicable, and every indicable group with torsion gives such an example. Conversely, locally indicable groups need not be indicable. For instance, every non-trivial finitely generated subgroup of $\mathbb{Q}$ is infinite cyclic, but there is no non-trivial group homomorphism $\mathbb{Q} \to \mathbb{Z}$.

(5) Every group G that fits into an exact sequence of groups of the form $1 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 1$ with N and Q locally indicable, is again locally indicable. Indeed, if H is a non-trivial finitely generated subgroup of G, then either $H \leq i(N) \cong N$, in which case H admits a surjective homomorphism onto $\mathbb{Z}$, or $p(H)$ is a non-trivial finitely generated subgroup of Q, in which case H admits a surjective homomorphism onto $\mathbb{Z}$ that factors through $H \xrightarrow{p_{|H}} p(H)$.

In particular, *free-by-{infinite cyclic} groups*, i.e., groups G that fit into an exact sequence $1 \to F \to G \to \mathbb{Z} \to 1$ with F a finitely generated free group, are locally indicable groups by (1) and (2).

(6) Except for the fundamental group of the projective plane, which is not locally indicable because it has torsion, the *fundamental groups of connected closed surfaces with genus* $g \geq 1$ are locally indicable. Here, we distinguish the fundamental groups S_g of orientable closed surfaces of genus $g \geq 1$, which admit the presentations

$$S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot \ldots \cdot [a_g, b_g] \rangle,$$

and the fundamental groups of non-orientable closed surfaces of genus $g \geq 2$, which admit the presentations

$$\mathfrak{S}_g = \langle a_1, \ldots, a_g \mid a_1^2 \cdot \ldots \cdot a_g^2 \rangle.$$

Every group G in this family contains a normal free subgroup F such that G/F is infinite cyclic, as a consequence of the fact that their infinite index subgroups are free (cf. [HKS72]) and that their abelianizations contain an infinite cyclic summand. Hence, G fits into an exact sequence $1 \to F \to G \to \mathbb{Z} \to 1$ with F a (non-necessarily finitely generated) free group, and (5) applies.

(7) More generally, *torsion-free one-relator groups*, i.e., torsion-free groups that admit a presentation with only one relation, are locally indicable, a result proved by S.D. Brodskii in [Bro84].

(8) In addition to the operations described in (2) and (5), the family of locally indicable groups has good closure properties. For instance, it is closed under cartesian products, direct sums, restricted standard wreath products, subdirect products, free products or directed unions (see, for instance, [Sán08, Proposition 2.6, Corollary 2.7, Corollary 2.9 & Proposition 2.11], where these and other closure properties are discussed).

□

Another family of groups that is deeply related to locally indicable groups is that of left (right) orderable groups.

Definition 3.4.16. Let G be a group. G is a *left orderable group* if there exists a total order $\leq$ on G such that, for all g, h with $g \leq h$ and for all $k \in G$, we have $kg \leq kh$. In this case we say that $\leq$ is a *left order* on G or that $(G, \leq)$ is a *left-ordered group*.

Similarly, we define *right orderable groups*, and we say that G is *orderable* if it admits a total order which is invariant under both left and right multiplication.

If we have a left-ordered group $(G, \leq)$, then the relation $\leq'$ defined by $g \leq' h$ if and only if $h^{-1} \leq g^{-1}$ defines a right order on G, so that every left orderable group is right orderable and viceversa. We are interested in a particular kind of ordering.

Definition 3.4.17. Let G be a group. A left order $\leq$ is *Conradian* (or *of Conrad type*) if for all $g, h \in G$ with $g, h > e$, we have that $hg^2 > g$. Similarly, a right order $\leq$ is Conradian if for all $g, h \in G$ with $g, h > e$, we have that $g^2h > g$.

Conradian orders can actually be defined in several equivalent ways, for instance involving the convex subgroups of G. The reader may consult [Nav10, Section 3.3], [RR02, Section 2] or [Sán08, Section 2.4] to see these different characterizations. We also recommend [DNR16] for a deeper look to left (right) orderable groups.

The following result, which gives the relation between locally indicable groups and left (right) orderable groups is usually attributed to S.D.Brodskii ([Bro84]), and different proofs of this fact can be found for instance in [Nav10, Proposition 3.11 & Proposition 3.16] and [RR02, Theorem 4.1].

Theorem 3.4.18. *The following are equivalent for a group G.*

(i) *G is locally indicable.*

(ii) *G admits a Conradian left order.*

(iii) *G admits a Conradian right order.*

In particular, this implies that we have the following sequence of strict containments between families of groups.

Orderable groups	$\subsetneq$	Locally indicable groups	$\subsetneq$	Left (right) orderable groups

On the one hand, if $(G, \leq)$ is ordered and $g, h > e$, then by left invariance $hg > h > e$ and using right invariance $hg^2 > g$, what means that $\leq$ is Conradian and hence G is a locally indicable group. The containment is strict, because for instance the fundamental group of the Klein bottle, $\mathfrak{S}_2$, is locally indicable (see Example 3.4.15(6)) but cannot be ordered. This can be seen using its alternative presentation $\langle c, d : cdc^{-1}d\rangle$ (which is obtained from the original via $c = ab^{-1}a^{-1}, d = ab$, and from which the original is recovered via $a = cd, b = d^{-1}c^{-1}d$). If we had a bi-invariant order $\leq$ on $\langle c, d : cdc^{-1}d\rangle$ with $d > e$, then from the right and left invariance we would obtain $d^{-1} = cdc^{-1} > cc^{-1} = e$, and hence $e > d$, a contradiction (and analogously if we had $d < e$).

On the other hand, the right containment is a consequence of the previous theorem, and an example of a right orderable not locally indicable group is given by G.M. Bergman in [Berg91].

As we mentioned at the beginning of the subsection, we want to introduce a division ring of fractions related to crossed products $E * G$ where E is a division ring and G is a locally indicable group, and hence it is particularly important to know whether such rings are domains or not. In [Hig40], G.Higman gave a positive answer to this question and also studied the units in $E * G$. It turns out that the same results hold true for left orderable groups, so that we have the following (cf. [Sán08, Proposition 4.8]).

Proposition 3.4.19. *Let $E * G$ be a crossed product of a division ring E and a left orderable group G. Then $E * G$ is a domain and $(E * G)^\times = E^\times G$.*

Actually, in the same reference it is proved that we can even substitute E by any domain. I. Kaplansky conjectured that the same should hold for group rings $K[G]$, where K is a (commutative) field and G is any torsion-free group. These conjectures are respectively known as Kaplansky's zero-divisor conjecture and Kaplansky's unit conjecture.

The next property is also the motivation for introducing the Hughes-free division ring of fractions. For a crossed product $E * G$, we write π_G to denote the composition $\pi_G : E^\times G \to E^\times G/E^\times \cong G$.

Lemma 3.4.20. *Let $E * G$ be a crossed product of a division ring E and a locally indicable group G. For every non-trivial finitely generated subgroup $H \leq G$, for every normal subgroup $N \lhd H$ such that H/N is infinite cyclic and for every $x \in E^\times H$ such that $H/N = \langle N\pi_H(x)\rangle$, the powers of x are left (and right) $E * N$-linearly independent.*

Proof. Let H be a non-trivial finitely generated subgroup of G, $N \lhd H$ with H/N infinite cyclic and $x \in E^\times H$ such that $H/N = \langle N\pi_H(x)\rangle$. This latter property implies that $T = \{\pi_H(x)^n : n \in \mathbb{Z}\}$ is a transversal of N in H, so as in the proof of Proposition 3.4.11 we can see that $E * H = \bigoplus_{n\in\mathbb{Z}}(E * N)x^n$. With this $\mathbb{Z}$-grading, $E * H$ is a crossed product $(E * N) * \mathbb{Z}$ with trivial twisting $\alpha(s^n, s^m) = x^n x^m x^{-(n+m)} = 1_{E*H}$ and action σ given by $\sigma_{s^n}(y) = x^n y x^{-n} = \tau^n(y)$, where $\tau = \sigma_s$ is the automorphism of $E * N$ given by left conjugation by x. Hence, as in the proof of Proposition 3.4.10, there exists an $E * N$-isomorphism $E * H \cong (E * N)[t^{\pm 1}; \tau]$ sending $x \mapsto t$. From here we also see that the powers of x are right $E * N$-linearly independent. $\square$

The Hughes-free division ring of fractions extends this property of linear independence. To avoid an overload of notation, for a division $E * G$-ring of fractions $\mathcal{D}$ and a subgroup $H \leq G$, we use $\mathcal{D}_H$ to denote the division closure of $E * H$ in $\mathcal{D}$. Usually, both the crossed product and the division ring of fractions are fixed, so that this should not cause misunderstandings.

Definition 3.4.21. Let $E * G$ be a crossed product of a division ring E and a locally indicable group G, and let $\mathcal{D}$ be a division $E * G$-ring of fractions. We say that $\mathcal{D}$ is a *Hughes-free division $E * G$-ring of fractions* (or simply *Hughes-free*) if for every non-trivial finitely generated subgroup $H \leq G$, for every $N \lhd H$ with H/N infinite cyclic and for every $x \in E^\times H$ such that $H/N = \langle N\pi_H(x)\rangle$, the powers of x are left $\mathcal{D}_N$-linearly independent.

An important remark is that it is not necessary to check linear independence for every $x \in E^\times H$ in the conditions of the definition. It suffices to check it for just one element, and in this sense, we can always fix a basis $\{u_g : g \in G\}$ with the usual properties $\alpha(e, g) = \alpha(g, e) = 1_E$ and $u_e = 1_{E*G}$ (Remark 3.4.9), and prove the result for a representative u_h such that $H/N = \langle Nh\rangle$.

Lemma 3.4.22. *Let $E*G$ be a crossed product of a division ring E and a locally indicable group G, and let $\mathcal{D}$ be a division $E * G$-ring of fractions. Then $\mathcal{D}$ is Hughes-free if and only if for every non-trivial finitely generated subgroup $H \leq G$ and for every $N \lhd H$ with H/N infinite cyclic, there exists an element $x \in E^\times H$ such that $H/N = \langle N\pi_H(x)\rangle$ and whose powers are left $\mathcal{D}_N$-linearly independent.*

Proof. Assume that the crossed product is given in the basis $\{u_g : g \in G\}$, let $x = ru_h$ be such that $H/N = \langle Nh\rangle$, and consider another $y = su_k \in E^\times H$ such that $H/N = \langle Nk\rangle$. Since there are only two generators of $\mathbb{Z}$ we either have $Nk = Nh$ or $Nk = Nh^{-1}$. In the first case, there exists $n \in N$ such that $k = nh$, and hence

$$y = su_k = su_{nh} = s\alpha(n, h)^{-1}u_n u_h = s\alpha(n, h)^{-1}\sigma_n(r^{-1})u_n x = ux$$

for some unit $u \in E^\times N$. Similarly, there exists a unit $u^{(m)} \in E^\times N$ such that $y^m = u^{(m)}x^m$. Hence, if we had an expression $\sum d_m y^m = 0$ for some $d_m \in \mathcal{D}_N$, we would have $\sum d_m u^{(m)} x^m = 0$. Since $d_m u^{(m)} \in \mathcal{D}_N$, the left $\mathcal{D}_N$-linear independence of the powers of x implies $d_m u^{(m)} = 0$ for every m, and since $u^{(m)}$ is a unit, $d_m = 0$. Hence, the powers of y are independent.

In the second case, there exists $n \in N$ such that $k = nh^{-1}$. Since $x^{-1} \in E^\times u_{h^{-1}}$, there exists a unit $v \in E$ such that $u_{h^{-1}} = vx^{-1}$, and hence

$$y = su_{nh^{-1}} = s\alpha(n, h^{-1})^{-1}u_n u_{h^{-1}} = s\alpha(n, h^{-1})^{-1}\sigma_n(v)u_n x^{-1} = wx^{-1}$$

for some unit $w \in E^\times N$, and proceeding as before there exists $w^{(m)} \in E^\times N$ with $y^{(m)} = w^{(m)}x^{-m}$ so we can reason as above to see that the powers of y must be left $\mathcal{D}_N$-linearly independent. □

I. Hughes proved in [Hug70] that, as it happens with the universal division ring of fractions, the Hughes-free division ring of fractions for $E * G$, if it exists, is unique up to $E * G$-isomorphism. Since the original proof is very condensed, we recommend [DHS04] (or [Sán08, Hughes' Theorem I]) for a different and more detailed proof of this fact. We also give an alternative proof in Chapter 4, Theorem 4.3.14, using in a different way the methods developed in [DHS04].

Theorem 3.4.23. *Let $E * G$ be a crossed product of a division ring E and a locally indicable group G. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two Hughes-free division $E * G$-rings of fractions, then there exists a unique $E * G$-isomorphism $\varphi : \mathcal{D}_1 \to \mathcal{D}_2$.*

It is still an open question whether every crossed product $E * G$ as before admits a Hughes-free division ring of fractions. In Chapter 4 we answer the question in the positive for group rings $K[G]$ over a (commutative) field K of characteristic zero, and we explore the relation between the Hughes-free division ring of fractions and the universal one in Chapter 5 (this was done in [JL20]). Nevertheless, for the groups that will appear in the next section, this problem was solved in full generality. In order to state the result properly, we first introduce a few definitions and results.

Definition 3.4.24. We say that a locally indicable group G is *Hughes-free embeddable* if for every division ring E and for every crossed product $E * G$, there exists a Hughes-free division $E * G$-ring of fractions.

Free groups, orderable groups or right orderable amenable groups are examples of Hughes-free embeddable groups (cf. [Sán08, Examples 5.6]. For free groups, one can also consult [Lew74, Proposition 6]). Moreover, I.Hughes proved in [Hug72] that Hughes-free embeddability is closed under extensions (cf. [Sán08, Hughes' Theorem II] for another proof of this fact).

Theorem 3.4.25. *Let G be a locally indicable group with a normal subgroup N such that G/N is locally indicable. If N and G/N are Hughes-free embeddable, then so is G.*

This, together with [Sán08, Example 6.19 & Proposition 6.23], gives us the next result. In Section 5.3 we show that it is always the case that if a Hughes-free and a universal division ring of fractions for $E * G$ exist, they must be $E * G$-isomorphic, hence giving another argument for the final statement of the proposition.

Proposition 3.4.26. *Let G be a group obtained as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is a free group. Then, for every division ring E and any crossed product $E * G$, there exists a Hughes-free division $E * G$-ring of fractions $\mathcal{D}$. Moreover, if there exists a universal division $E * G$-ring, then it isomorphic to $\mathcal{D}$.

Proof. As a consequence of the previous examples and results, G is Hughes-free embeddable. For the final statement, apply [Sán08, Example 6.19 & Proposition 6.23] to the subnormal series $1 \trianglelefteq F \trianglelefteq G$. $\square$

In fact, the existence of a universal division ring of fractions for the crossed products appearing in the previous proposition was already shown in [Jai20B]. In Section 3.5 we give an independent proof of this result by showing that they are actually pseudo-Sylvester domains, what gives in addition the precise set of matrices becoming invertible over it.

Another important remark here is that for every crossed product $E * G$ of a division ring and a locally indicable group, there is a canonical candidate to be the Hughes-free division $E * G$-ring of fractions (cf. [Grä20]), and that has to do with the space of Malcev-Neumann series for left orderable groups.

Let E be a division ring, G a left orderable group, and let $\leq$ be a left order on G. Consider the set $E((G, \leq))$ of formal power series

$$x = \sum_{g\in G} u_g r_g, \quad \text{with } r_g \in E,$$

whose support $\operatorname{supp}(x) = \{g \in G : r_g \neq 0\}$ is well-ordered with respect to $\leq$.

Malcev ([Mal48]) and Neumann ([Neu49]) proved independently that, if $(G, \leq)$ is ordered, the natural sum and product of series are well-defined, and $K((G, \leq))$ for a (commutative) field K is a division ring in which $K[G]$ embeds. If G is just left orderable, $E((G, \leq))$ is not a ring, but it is still a right E-vector space. Assume now that we have a crossed product $E * G$. If $x \in E((G, \leq))$ and $ru_h \in E^\times G$, we can define $ru_h \cdot x$ by just extending the product defined in $E * G$. In this way, the support of the element obtained is $\{hg : g \in \operatorname{supp}(x)\}$, which by left compatibility of $\leq$ is well-ordered with least element hg_0, where g_0 is the least element of $\operatorname{supp}(x)$. Thus, left multiplication by ru_h defines an element of $\operatorname{End}(E((G, \leq)))$, and this can be linearly extended to any element in $E * G$ since subsets and finite unions of well-ordered sets are again well-ordered. By construction, this identification is compatible with the product in $E * G$, and therefore we can see $E * G$ as a subring of $\operatorname{End}(E((G, \leq)))$. We recommend the reader to consult [Grä20, Section 7] for a detailed explanation and further properties of this embedding.

The following is a combination of [Grä20, Theorem 8.1 & Corollary 8.3].

Theorem 3.4.27. *Let E be a division ring and G a locally indicable group. If there exists a Hughes-free division $E*G$-ring of fractions, then it is isomorphic to the division closure of $E * G$ inside* $\operatorname{End}(E((G, \leq)))$, *where $\leq$ is any Conradian left order on G.*

We finish the section introducing a characterization of Hughes-free division rings of fractions in terms of the Sylvester matrix rank function that they induce on $E*G$. As we did for the Hughes-free definition, in order to ease the notation, for every $\mathrm{rk} \in \mathbb{P}(E * G)$ and every $H \leq G$, we use $\mathrm{rk}_H \in \mathbb{P}(E * H)$ to denote the restriction of rk to $E * H$.

Let H be a group and $N \triangleleft H$ a normal subgroup such that H/N is infinite cyclic, and consider a crossed product $E * H$ with a rank function $\mathrm{rk} \in \mathbb{P}(E * H)$. We have seen in Lemma 3.4.20 that, for every $x \in E^\times H$ such that $\langle N\pi_H(x)\rangle = H/N$, there exists an $E*N$-isomorphism $\varphi_x : E*H \to E*N[t^{\pm 1};\tau_x]$, where $x \mapsto t$ and τ_x is the automorphism of $E*N$ given by left conjugation by x. Since x is a unit in $E*H$ and rk_N is the restriction of rk to $E*N$ we have that, for every $n\times m$ matrix A over $E*N$,

$$\mathrm{rk}_N(\tau_x(A)) = \mathrm{rk}((xI_n)A(x^{-1}I_m)) = \mathrm{rk}(A) = \mathrm{rk}_N(A).$$

Hence, rk_N is τ_x-compatible. Therefore, it makes sense to talk about the natural transcendental extension of rk_N to $E*N[t^{\pm 1};\tau_x]$. In this case, $\varphi_x^\sharp(\widetilde{\mathrm{rk}}_N)$ defines a rank function on $E*H$. Moreover, we show in the next lemma that the resulting rank function on $E*H$ does not depend on the choice of x (and hence of the isomorpshim).

Lemma 3.4.28. *Let H be a group, consider a crossed product $E*H$ and let $\mathrm{rk} \in \mathbb{P}(E*H)$. Assume that $N \triangleleft H$ is a normal subgroup with H/N infinite cyclic and that $x_1, x_2 \in E^\times H$ satisfy $H/N = \langle N\pi_H(x_1)\rangle = \langle N\pi_H(x_2)\rangle$. Consider the induced $E*N$-isomorphisms $\varphi_{x_i} : E*H \to E*N[t_i^{\pm 1};\tau_{x_i}]$ with $\varphi_{x_i}(x_i) = t_i$ and where τ_{x_i} is the automorphism of $E*N$ given by conjugation by x_i. If $\widetilde{\mathrm{rk}}_N$ denotes both the natural transcendental extension of rk_N to $E*N[t_1^{\pm 1};\tau_{x_1}]$ and to $E*N[t_2^{\pm 1};\tau_{x_2}]$, respectively, then $\varphi_{x_1}^\sharp(\widetilde{\mathrm{rk}}_N) = \varphi_{x_2}^\sharp(\widetilde{\mathrm{rk}}_N)$.*

Proof. Consider the $E*N$-isomorphism

$$\varphi_{x_1}\circ\varphi_{x_2}^{-1} : E*N[t_2^{\pm 1};\tau_{x_2}] \to E*N[t_1^{\pm 1};\tau_{x_1}]$$

and observe that, if $\widetilde{\mathrm{rk}}_N$ is the natural extension of rk_N to $E*N[t_1^{\pm 1};\tau_{x_1}]$, the formula $\mathrm{rk}' = (\varphi_{x_1}\circ\varphi_{x_2}^{-1})^\sharp(\widetilde{\mathrm{rk}}_N)$ defines a Sylvester matrix rank function on $E*N[t_2^{\pm 1};\tau_{x_2}]$.

Note also that we have seen in the proof of Lemma 3.4.22 that for such x_1, x_2, there exists a unit u in $E^\times N$ such that $x_2 = ux_1^{\pm 1}$. Therefore, we must care essentially about the following two situations.

Case 1: $x_2 = ux_1$.

Let $u_{(k)} \in E^\times N$ be such that $x_2^k = u_{(k)}x_1^k$. Note then that, since φ_{x_1} is a homomorphism, we have $(ut_1)^k = \varphi_{x_1}((ux_1)^k) = \varphi_{x_1}(u_{(k)}x_1^k) = u_{(k)}t_1^k$, and that for every $a \in E*N$ and $k\in\mathbb{Z}$, $\tau_{x_1}^k$ and $\tau_{x_2}^k$ are related by

$$\tau_{x_2}^k(a) = x_2^k a x_2^{-k} = u_{(k)}x_1^k a x_1^{-k}u_{(k)}^{-1} = u_{(k)}\tau_{x_1}^k(a)u_{(k)}^{-1}.$$

Let $p(t_2) = \sum a_i t_2^i$ be a polynomial in $E*N[t_2;\tau_{x_2}]$, and observe that

$$\begin{aligned}\varphi_{x_1}\circ\varphi_{x_2}^{-1}(p(t_2)) &= \varphi_{x_1}\left(\textstyle\sum a_i x_2^i\right) = \varphi_{x_1}\left(\textstyle\sum a_i u_{(i)}x_1^i\right)\\ &= \textstyle\sum a_i u_{(i)} t_1^i = \sum a_i (ut_1)^i =: p'(t_1)\end{aligned}$$

Hence, $\mathrm{rk}'(p(t_2)) = \widetilde{\mathrm{rk}}_N(p'(t_1)) = \lim\limits_{k\to\infty} \dfrac{\mathrm{rk}_N(\phi_k^{p'(t_1)})}{k}$. Since

$$\begin{aligned} t_2^n\cdot p(t_2) &= \textstyle\sum_i t_2^n a_i t_2^i = \sum_i \tau_{x_2}^n(a_i)t_2^{n+i},\\ (ut_1)^n\cdot p'(t_1) &= \textstyle\sum_i u_{(n)}t_1^n a_i (ut_1)^i = \sum_i u_{(n)}\tau_{x_1}^n(a_i)u_{(n)}^{-1}u_{(n)}t_1^n(ut_1)^i\\ &= \textstyle\sum_i \tau_{x_2}^n(a_i)(ut_1)^{n+i}.\end{aligned}$$

we observe that the matrix associated to $\phi_k^{p(t_2)}$ with respect to the canonical bases $\{t_2^j + E * N[t_2;\tau_{x_2}]t_2^k\}_{j=0}^{k-1}$ in both the domain and codomain coincides with the matrix associated to $\phi_k^{p'(t_1)}$ with respect to the corresponding bases $\{u_{(j)}t_1^j + E * N[t_1;\tau_{x_1}]t_1^k\}_{j=0}^{k-1}$ in the domain and codomain, and hence $\mathrm{rk}_N(\phi_k^{p(t_2)}) = \mathrm{rk}_N(\phi_k^{p'(t_1)})$ for every k. Thus,

$$\mathrm{rk}'(p(t_2)) = \lim_{k\to\infty} \frac{\mathrm{rk}_N(\phi_k^{p'(t_1)})}{k} = \lim_{k\to\infty} \frac{\mathrm{rk}_N(\phi_k^{p(t_2)})}{k}$$

In particular, the latter limit exists. The same reasoning for matrices over $E * N[t_2;\tau_{x_2}]$ shows that the natural extension of rk_N to $E * N[t_2;\tau_{x_2}]$ coincides with the restriction of rk'. By Remark 1.4.19 this implies that the natural extension of rk_N to $E * N[t_2^{\pm 1};\tau_{x_2}]$ equals rk'. Hence, as rank functions on $E * H$, we have that $\varphi_{x_2}^\sharp(\widetilde{\mathrm{rk}}_N) = \varphi_{x_1}^\sharp(\widetilde{\mathrm{rk}}_N)$, where the left and right hand $\widetilde{\mathrm{rk}}_N$ are, respectively, ranks on $E * N[t_2^{\pm1};\tau_{x_2}]$ and $E * N[t_1^{\pm1};\tau_{x_1}]$.

Case 2: $x_2 = x_1^{-1}$.

Observe that in this case $\tau_{x_2} = \tau_{x_1}^{-1}$ and that, for a polynomial $p(t_2) = \sum_{i=0}^n a_i t_2^i$ in $E * N[t_2;\tau_{x_2}]$, we have

$$\varphi_{x_1} \circ \varphi_{x_2}^{-1}(p(t_2)) = \varphi_{x_1}(\textstyle\sum_{i=0}^n a_i x_2^i) = \varphi_{x_1}(\sum_{i=0}^n a_i x_1^{-i}) = \sum_{i=0}^n a_i t_1^{-i}$$

and hence $\mathrm{rk}'(p(t_2)) = \widetilde{\mathrm{rk}}_N(\sum_{i=0}^n a_i t_1^{-i}) = \widetilde{\mathrm{rk}}_N(\sum_{i=0}^n a_{n-i} t_1^i)$, where the latter equality is obtained by multiplying the unit t_1^n.

Set $p'(t_1) = \sum_{i=0}^n a_{n-i}t_1^i$. Let us illustrate the form of the matrix associated to $\phi_k^{p'(t_1)}$ with respect to the canonical bases in the domain and codomain for low degree $n = 2$. In this case, $p'(t_1) = a_2 + a_1t_1 + a_0t_1^2$, and hence the aforementioned matrix for $k = 5$ is the 5×5 matrix

$$\left[\begin{array}{cc|ccc} a_2 & a_1 & a_0 & 0 & 0 \\ 0 & \tau_{x_1}(a_2) & \tau_{x_1}(a_1) & \tau_{x_1}(a_0) & 0 \\ 0 & 0 & \tau_{x_1}^2(a_2) & \tau_{x_1}^2(a_1) & \tau_{x_1}^2(a_0) \\ 0 & 0 & 0 & \tau_{x_1}^3(a_2) & \tau_{x_1}^3(a_1) \\ 0 & 0 & 0 & 0 & \tau_{x_1}^4(a_2) \end{array}\right]$$

By interchanging rows and columns, the previous matrix is equivalent to

$$\left[\begin{array}{cc|ccc} 0 & 0 & \tau_{x_1}^4(a_2) & 0 & 0 \\ 0 & 0 & \tau_{x_1}^3(a_1) & \tau_{x_1}^3(a_2) & 0 \\ 0 & 0 & \tau_{x_1}^2(a_0) & \tau_{x_1}^2(a_1) & \tau_{x_1}^2(a_2) \\ 0 & \tau_{x_1}(a_2) & 0 & \tau_{x_1}(a_0) & \tau_{x_1}(a_1) \\ a_2 & a_1 & 0 & 0 & a_0 \end{array}\right]$$

and since rk_N is τ_{x_1}-compatible and $\tau_{x_2} = \tau_{x_1}^{-1}$, the rk_N-rank of the previous matrix is (by applying $\tau_{x_1}^{-4} = \tau_{x_2}^4$) the rk_N-rank of

$$\left[\begin{array}{cc|ccc} 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & \tau_{x_2}(a_1) & \tau_{x_2}(a_2) & 0 \\ 0 & 0 & \tau_{x_2}^2(a_0) & \tau_{x_2}^2(a_1) & \tau_{x_2}^2(a_2) \\ 0 & \tau_{x_2}^3(a_2) & 0 & \tau_{x_2}^3(a_0) & \tau_{x_2}^3(a_1) \\ \tau_{x_2}^4(a_2) & \tau_{x_2}^4(a_1) & 0 & 0 & \tau_{x_2}^4(a_0) \end{array}\right].$$

On the other hand, since $p(t_2) = a_0 + a_1 t_2 + a_2 t_2^2$, the matrix associated to $\phi_k^{p(t_2)}$ with respect to the canonical bases is given by

$$\left[\begin{array}{cc|ccc} a_0 & a_1 & a_2 & 0 & 0 \\ 0 & \tau_{x_2}(a_0) & \tau_{x_2}(a_1) & \tau_{x_2}(a_2) & 0 \\ 0 & 0 & \tau_{x_2}^2(a_0) & \tau_{x_2}^2(a_1) & \tau_{x_2}^2(a_2) \\ 0 & 0 & 0 & \tau_{x_2}^3(a_0) & \tau_{x_2}^3(a_1) \\ 0 & 0 & 0 & 0 & \tau_{x_2}^4(a_0) \end{array}\right].$$

The previous two matrices coincide in the last $3 = k - n$ columns. In the general case, what happens is that after rearranging rows and columns and applying $\tau_{x_1}^{-k}$, computing $\operatorname{rk}_N(\phi_k^{p'(t_1)})$ amounts to computing the rank of a matrix whose last $k-n$ columns coincide with the last $k-n$ columns of the matrix associated to $\phi_k^{p(t_2)}$ with respect to the canonical bases. In other words, there exist a matrix A of size $k \times (k-n)$ and matrices A_1, A_2 of sizes $k \times n$ over $E * N$ such that

$$\operatorname{rk}_N(\phi_k^{p'(t_1)}) = \operatorname{rk}_N(A_1 \quad A) \text{ and } \operatorname{rk}_N(\phi_k^{p(t_2)}) = \operatorname{rk}_N(A_2 \quad A)$$

Since, using the properties of Sylvester matrix rank functions, one has

$$\operatorname{rk}_N(A) \leq \operatorname{rk}_N(A_i \quad A) \leq \operatorname{rk}_N(A) + n,$$

we deduce that $|\operatorname{rk}_N(\phi_k^{p'(t_1)}) - \operatorname{rk}_N(\phi_k^{p(t_2)})| \leq n$. Consequently,

$$\lim_{k\to\infty} \frac{|\operatorname{rk}_N(\phi_k^{p'(t_1)}) - \operatorname{rk}_N(\phi_k^{p(t_2)})|}{k} = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{\operatorname{rk}_N(\phi_k^{p'(t_1)}) - \operatorname{rk}_N(\phi_k^{p(t_2)})}{k} = 0$$

and, since $\lim_{k\to\infty} \frac{\operatorname{rk}_N(\phi_k^{p'(t_1)})}{k} = \widetilde{\operatorname{rk}}_N(p'(t_1))$ also exists, we conclude that

$$\operatorname{rk}'(p(t_2)) = \widetilde{\operatorname{rk}}_N(p'(t_1)) = \lim_{k\to\infty} \frac{\operatorname{rk}_N(\phi_k^{p'(t_1)})}{k} = \lim_{k\to\infty} \frac{\operatorname{rk}_N(\phi_k^{p(t_2)})}{k}$$

Reasoning similarly for matrices over $E * N[t_2; \tau_{x_2}]$, this shows as in the previous case that the natural extension of rk_N to $E * N[t_2^{\pm 1}; \tau_{x_2}]$ equals rk'. Hence, as rank functions on $E * H$, we have again that $\varphi_{x_2}^\sharp(\widetilde{\operatorname{rk}}_N) = \varphi_{x_1}^\sharp(\widetilde{\operatorname{rk}}_N)$.

□

Definition 3.4.29. We say that the (unique) Sylvester matrix rank function on $E * H$ constructed in Lemma 3.4.28 is the *natural transcendental extension of* rk_N *to* $E * H$.

Observe that we used that rk_N is induced from a rank on $E * H$ to ensure its compatibility with the automorphism τ_{x_1}. However, it may not be true that the rank function $\varphi_{x_1}^\sharp(\widetilde{\operatorname{rk}}_N)$ in the conclusion of Lemma 3.4.28 coincides with the original $\operatorname{rk} \in \mathbb{P}(E * H)$. We are interested precisely in the case when $\operatorname{rk} = \varphi_{x_1}^\sharp(\widetilde{\operatorname{rk}}_N)$, what justifies our definition of Hughes-free rank.

Definition 3.4.30. Let $E * G$ be a crossed product of a division ring E and a locally indicable group G. A Sylvester matrix rank function $\mathrm{rk} \in \mathbb{P}(E * G)$ is *Hughes-free* if, for every non-trivial finitely generated subgroup $H \leq G$ and for every $N \lhd H$ with H/N infinite cyclic, rk_H is the natural transcendental extension of rk_N.

The next result states the relation between Hughes-free rank functions coming from a division ring of fractions and Hughes-free division ring of fractions.

Proposition 3.4.31. *Let $E * G$ be a crossed product of a division ring E and a locally indicable group G, and let $\mathcal{D}$ be a division $E*G$-ring of fractions. Then $\mathcal{D}$ is Hughes-free if and only if $\mathrm{rk}_{\mathcal{D}}$, as a Sylvester matrix rank function on $E * G$, is Hughes free.*

Proof. Let H be a non-trivial finitely generated subgroup of G, and let $N \lhd H$ be such that H/N is infinite cyclic. Let us also denote by rk_N and rk_H, respectively, the restrictions of $\mathrm{rk}_{\mathcal{D}}$ to $E * N$ and $E * H$.

Now, fix $x \in E^\times H$ such that $H/N = \langle N\pi_H(x)\rangle$, so that if τ_x is the automorphism of $E * N$ given by $\tau_x(y) = xyx^{-1}$ for $y \in E * N$, we have the isomorphism $\varphi_x : E * H \to (E * N)[t^{\pm 1}; \tau_x]$ that acts as the identity on $E * N$ and sends $x \mapsto t$. Let S be the subring of $\mathcal{D}_H$ generated by $\mathcal{D}_N$, x and x^{-1}.

By Lemma 3.3.4, τ_x extends to an automorphism, also denoted τ_x, of $\mathcal{D}_N$, so we can form the skew Laurent polynomial ring $\mathcal{D}_N[t^{\pm 1}; \tau_x]$, together with an embedding $\iota_1 : E * N[t^{\pm 1}; \tau_x] \hookrightarrow \mathcal{D}_N[t^{\pm 1}; \tau_x]$ acting as the identity on $E * N$. Since $\mathcal{D}_N$ is a division ring and τ_x is an automorphism, we can consider its Ore division ring $\mathcal{D}_N(t; \tau)$ (see Example 3.1.7) together with an embedding $\iota_2 : \mathcal{D}_N[t^{\pm 1}; \tau] \hookrightarrow \mathcal{D}_N(t; \tau)$. Therefore, we have,

$$\begin{tikzcd} E * H \arrow[r, hook, "j_1"] \arrow[d, "\varphi_x"', "\cong"] & S \arrow[r, hook, "j_2"] & \mathcal{D}_H \\ E * N[t^{\pm 1}; \tau_x] \arrow[r, hook, "\iota_1"] & \mathcal{D}_N[t^{\pm 1}; \tau_x] \arrow[r, hook, "\iota_2"] & \mathcal{D}_N(t; \tau_x) \end{tikzcd}$$

Moreover, note that since rk_N has $\mathcal{D}_N$ as epic division envelope, Proposition 3.1.20 (3.) tells us that $(\mathcal{D}_N(t; \tau_x), \iota_2 \circ \iota_1)$ is the epic division envelope of $\widetilde{\mathrm{rk}}_N$ as a rank function on $E * N[t^{\pm 1}; \tau_x]$, i.e., $\widetilde{\mathrm{rk}}_N = (\iota_2 \circ \iota_1)^\sharp(\mathrm{rk}_{\mathcal{D}_N(t;\tau_x)})$.

Assume first that $\mathcal{D}$ is Hughes-free. Then the powers of x are left $\mathcal{D}_N$-linearly independent and thus the isomorphism φ_x extends to an isomorphism $S \xrightarrow{\cong} \mathcal{D}_N[t^{\pm 1}; \tau_x]$. This gives us an embedding of $\mathcal{D}_N[t^{\pm 1}; \tau_x]$ into the division ring $\mathcal{D}_H$, from where the universal property of Ore localization (Proposition 3.1.4) gives us a ring homomorphism $\varphi : \mathcal{D}_N(t; \tau) \to \mathcal{D}_H$ so that the previous diagram is completed with commutative squares

$$\begin{tikzcd} E * H \arrow[r, hook, "j_1"] \arrow[d, "\varphi_x"', "\cong"] & S \arrow[r, hook, "j_2"] \arrow[d, "\cong"'] & \mathcal{D}_H \\ E * N[t^{\pm 1}; \tau_x] \arrow[r, hook, "\iota_1"] & \mathcal{D}_N[t^{\pm 1}; \tau_x] \arrow[r, hook, "\iota_2"] & \mathcal{D}_N(t; \tau_x). \arrow[u, "\varphi"'] \end{tikzcd}$$

Since $\mathcal{D}_N(t;\tau_x)$ is a division ring, φ must be injective, and since $E * H$ generates $\mathcal{D}_H$ as a division ring and $E * H \subseteq \operatorname{im}\varphi$, which is a division ring, φ must be surjective. Thus, φ is an $E * H$-isomorphism.

Since in a division ring there exists only one Sylvester matrix rank function we must have $\varphi^\sharp(\mathrm{rk}_{\mathcal{D}}) = \mathrm{rk}_{\mathcal{D}_N(t;\tau_x)}$, and hence,

$$\begin{aligned}\varphi_x^\sharp(\widetilde{\mathrm{rk}}_N) &= [\varphi_x^\sharp \circ (\iota_2 \circ \iota_1)^\sharp](\mathrm{rk}_{\mathcal{D}(t;\tau_x)}) = [\varphi_x^\sharp \circ (\iota_2 \circ \iota_1)^\sharp \circ \varphi^\sharp](\mathrm{rk}_{\mathcal{D}}) \\ &= (\varphi \circ \iota_2 \circ \iota_1 \circ \varphi_x)^\sharp(\mathrm{rk}_{\mathcal{D}}) = (j_2 \circ j_1)^\sharp(\mathrm{rk}_{\mathcal{D}}) = \mathrm{rk}_H .\end{aligned}$$

Thus, rk_H is the natural extension of rk_N, and since this is valid for every H, N and x, $\mathrm{rk}_{\mathcal{D}}$, as a rank function on $E * G$, is Hughes-free.

Assume conversely that $\mathrm{rk}_{\mathcal{D}}$, as a Sylvester matrix rank function on $E*G$, is a Hughes-free rank function, so that $\mathrm{rk}_H = \varphi_x^\sharp(\widetilde{\mathrm{rk}}_N)$. This implies that both $(\mathcal{D}_N(t;\tau_x), \iota_2 \circ \iota_1 \circ \varphi_x)$, and $(\mathcal{D}_H, j_2 \circ j_1)$ are epic division envelopes of rk_H. Hence, Corollary 3.1.17 tells us that there is an $E * H$-isomorphism $\psi : \mathcal{D}_H \to \mathcal{D}_N(t;\tau)$, so that the following commutes

$$\begin{tikzcd} E * H \arrow[r, hook, "j_2 \circ j_1"] \arrow[d, hook, "\varphi_x"', "\cong"] & \mathcal{D}_H \arrow[d, "\psi"] \\ E * N[t^{\pm 1};\tau_x] \arrow[r, hook, "\iota_2 \circ \iota_1"] & \mathcal{D}_N(t;\tau_x). \end{tikzcd}$$

Since $\iota_2 \circ \iota_1 \circ \varphi_x$ acts as the identity on $E * N$ and sends $x \mapsto t$, by commutativity we must have that ψ acts as the identity on $E * N$ and $\psi(x) = t$. We claim that ψ acts as the identity on $\mathcal{D}_N$. Indeed, recall first from Lemma 3.3.3 that $\mathcal{D}_N = \mathcal{D}_{E*N,\mathcal{D}_H}$, i.e., the division closure of $E * N$ inside $\mathcal{D}$ coincides with the division closure of $E * N$ in $\mathcal{D}_H$. Now, if we recover the construction $\mathcal{D}_N = \bigcup_{i=0}^{\infty} Q_i$ of Proposition 3.3.2, then we have that ψ acts as the identity on $Q_0 = E * N$. Assume inductively that ψ acts as the identity on Q_{i-1} for some $i \geq 1$. If $a \in Q_i$ is such that $a = b^{-1}$ for some $b \in Q_{i-1}$, then since ψ is a ring homomorphism, $\psi(a) = \psi(b)^{-1} = b^{-1} = a$. Therefore, ψ acts as the identity on the generators of Q_i, and hence, again because it is a homomorphism, it must act as the identity on Q_i. Since this holds for every i, ψ acts as the identity on $\mathcal{D}_N$. Therefore, if we have an expression $\sum d_i x^i = 0$ with $d_i \in \mathcal{D}_N$, then $0 = \psi\left(\sum d_i x^i\right) = \sum d_i t^i = \iota_2\left(\sum d_i t^i\right)$, and therefore $\sum d_i t^i = 0$ on $\mathcal{D}_N[t^{\pm 1};\tau]$, what can only happen if $d_i = 0$ for every i. Thus, the powers of x are left $\mathcal{D}_N$-linearly independent. Since this is valid for every H, N and x, $\mathcal{D}$ is Hughes-free. □

3.5 The crossed product $\mathfrak{F} * \mathbb{Z}$ of a fir $\mathfrak{F}$ and $\mathbb{Z}$

This section explores when a crossed product of the form $\mathfrak{F} * \mathbb{Z}$, where $\mathfrak{F}$ is a fir, is a (pseudo)-Sylvester domain, and contains the main results in [HL20]. Recall that firs are examples of Sylvester domains, so that the universal division $\mathfrak{F}$-ring of fractions $\mathcal{D}_{\mathfrak{F}}$ exists and coincides with the localization of $\mathfrak{F}$ with respect to the set of all full matrices. Troughout the section, $\mathcal{S}$ denotes a crossed product $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$ and $\mathcal{D}_{\mathfrak{F}}$ denotes the universal division ring of fractions of $\mathfrak{F}$.

To accomplish our goal, we study up to which extent these crossed products satisfy the conditions of Corollary 3.2.12 and Theorem 3.2.13, and in order to do that we first need a candidate for the universal division $\mathfrak{F}*\mathbb{Z}$-ring of fractions.

The following lemma tells us in particular that the crossed product structure $\mathcal{S} = \mathfrak{F}*\mathbb{Z}$ can always be extended to a crossed product structure $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, and that this ring is an Ore domain.

Lemma 3.5.1. *Let R be a (pseudo-)Sylvester domain with universal division R-ring of fractions $\mathcal{D}_R$. Then any crossed product structure $R * \mathbb{Z}$ extends to a crossed product $\mathcal{D}_R * \mathbb{Z}$. Moreover, $\mathcal{D}_R * \mathbb{Z}$ is an Ore domain and $\mathcal{Q}(\mathcal{D}_R * \mathbb{Z})$ is a division $R * \mathbb{Z}$-ring of fractions.*

Proof. First, we are going to see that every automorphism φ of R extends uniquely to an automorphism of $\mathcal{D}_R$. Let Σ denote the set of (stably) full matrices over R and notice that φ preserves Σ (i.e., $\varphi(\Sigma) = \Sigma$). Indeed, we already remarked after the definition of the stable rank that a matrix A is stably full if and only if $A \oplus I_s$ is full for every $s \geq 0$, and since any decomposition of a matrix as $B = CD$ gives the corresponding decomposition of $\varphi(B)$ and φ is an automorphism, we see that a matrix and its image have the same inner rank. Thus, the composition $R \xrightarrow{\varphi} R \hookrightarrow \mathcal{D}_R$ is a Σ-inverting embedding, and hence the universal property of universal localization gives us a unique ring homomorphism $\varphi : R_\Sigma = \mathcal{D}_R \to \mathcal{D}_R$ such that the diagram

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, "\varphi"', "\cong"] & \mathcal{D}_R \arrow[d, dotted, "\varphi"] \\
R \arrow[r, hook] & \mathcal{D}_R.
\end{tikzcd}$$

commutes. Since $\mathcal{D}_R$ is a division ring, φ is injective, while since $\mathcal{D}_R$ is generated by R as a division ring, φ is also surjective, and hence an automorphism of $\mathcal{D}_R$.

Since the extension is unique, by Proposition 3.4.6 the crossed product $R * \mathbb{Z}$ and the embedding $\iota_1 : R \hookrightarrow \mathcal{D}_R$ extend to a crossed product $\mathcal{D}_R * \mathbb{Z}$ and a ring homomorphism $\iota_2 : R * \mathbb{Z} \to \mathcal{D}_R * \mathbb{Z}$ such that the following commutes

$$\begin{tikzcd}
R \arrow[r, hook, "\iota_1"] \arrow[d, hook, "j_1"'] & \mathcal{D}_R \arrow[d, hook, "j_2"] \\
R * \mathbb{Z} \arrow[r, "\iota_2"] & \mathcal{D}_R * \mathbb{Z}.
\end{tikzcd}$$

Recall that $\mathcal{D}_R * \mathbb{Z}$ is constructed in the same basis that $R * \mathbb{Z}$ and that ι_2 sends $\sum r u_{s^i}$ to $\sum r u_{s^i}$. Hence, ι_2 is an embedding. Furthermore, either using that $\mathcal{D}_R * \mathbb{Z}$ can be seen as a skew Laurent polynomial ring (see Proposition 3.4.10) or the more general result of Tamari (see, for instance, [Kie20, Theorem 2.14]) taking into account that $\mathbb{Z}$ is amenable, $\mathcal{D}_R * \mathbb{Z}$ is an Ore domain and we can consider its Ore division ring $\mathcal{Q}(\mathcal{D}_R * \mathbb{Z})$.

Let S be any ring and $f, g : \mathcal{D}_R * \mathbb{Z} \to S$ ring homomorphisms with $f \circ \iota_2 = g \circ \iota_2$. This implies, on the one hand, that f and g coincide on the elements of the basis. On the other hand, this also implies that

$$f \circ j_2 \circ \iota_1 = f \circ \iota_2 \circ j_1 = g \circ \iota_2 \circ j_1 = g \circ j_2 \circ \iota_1,$$

and since ι_1 is epic, we deduce that $f \circ j_2 = g \circ j_2$, i.e., $f(d1_{\mathcal{D}_R * \mathbb{Z}}) = g(d1_{\mathcal{D}_R * \mathbb{Z}})$ for every $d \in \mathcal{D}_R$. Therefore, since f and g are ring homomorphisms, for every $x = \sum d u_{s^i} \in \mathcal{D}_R * \mathbb{Z}$ (see also Corollary 3.4.4(1)),

$$f(x) = \sum f(d1_{\mathcal{D}_R * \mathbb{Z}}) f(u_{s^i}) = \sum g(d1_{\mathcal{D}_R * \mathbb{Z}}) g(u_{s^i}) = g(x),$$

so that $f = g$. Therefore ι_2 is an epic embedding, and since the embedding $\mathcal{D}_R * \mathbb{Z} \hookrightarrow \mathcal{Q}(\mathcal{D}_R * \mathbb{Z})$ is also epic, the composition $R * \mathbb{Z} \hookrightarrow \mathcal{Q}(\mathcal{D}_R * \mathbb{Z})$ is epic. □

We denote $\mathcal{D}_{\mathcal{S}} = \mathcal{Q}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$, which will be the candidate to be the universal division $\mathcal{S}$-ring of fractions, and hence we are interested in studying its homological properties. Recall, for instance, that in Lemma 3.4.7 we explored the $\mathcal{S}$-module structure of the crossed product $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$.

The next lemma, applied to the case $R := \mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ and $S := \mathcal{S}$, will allow us later to restrict our attention to $\mathcal{S}$-submodules of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$:

Lemma 3.5.2. *Let R be a right Ore domain with right Ore division ring $\mathcal{Q}_r(R)$ and S a subring of R. Then every finitely generated S-submodule M of the left S-module $\mathcal{Q}_r(R)$ is isomorphic to a finitely generated S-submodule of R.*

Proof. Let M be generated as a left S-module by $x_1, \dots, x_m \in \mathcal{Q}_r(R)$. We find $p_i, q_i \in R$ such that $x_i = p_i q_i^{-1}$ for $i = 1, \dots, m$. If $m \geq 2$ we can use the Ore condition to find nonzero $a, b \in R$ such that $q_1 a = q_2 b$, and hence $x_1 = (p_1 a)(q_1 a)^{-1}$ and $x_2 = (p_2 b)(q_2 b)^{-1}$ can be expressed as fractions with common denominators. By repeatedly applying this procedure we produce $p_i', q \in R, q \neq 0$ such that $x_i = p_i' q^{-1}$ for all i.

We now consider the left S-submodule M' of R generated by $x_1 q, \dots, x_m q$. The map $f\colon M \to M'$ given by $y \mapsto yq$ is S-linear since $\mathcal{Q}_r(R)$ is associative and surjective since its image contains the generators. Finally, it is injective, since $\mathcal{Q}_r(R)$ is a division ring and hence $zq \neq 0$ for every $z \neq 0$. We conclude that f is an S-linear isomorphism. □

We shall also need some results on the behavior of the Tor functor with respect to flat modules and universal localizations. With respect to the first issue, we have the following result sometimes referred to as Shapiro's Lemma, that will allow us to relate $\mathrm{Tor}_*^{\mathcal{S}}$ with $\mathrm{Tor}_*^{\mathfrak{F}}$. This result can be found for instance in [Rot09, Corollary 10.61], but we give here an elementary proof.

Lemma 3.5.3. *Let R be a subring of S such that S is flat as a left R-module. Then, for any right R-module M, for any left S-module N and for any $n \geq 0$, we have*

$$\mathrm{Tor}_n^R(M, {}_R N) \cong \mathrm{Tor}_n^S(M \otimes_R S, N)$$

where ${}_R N$ denotes N considered as a left R-module.

Proof. Assume that we have a projective resolution for M

$$\ldots \to P_k \to \ldots \to P_0 \to M \to 0.$$

Since S is a flat left R-module, the following sequence is also exact of projective right S-modules, i.e., a projective resolution for $M \otimes_R S$

$$\ldots \to P_k \otimes_R S \to \ldots \to P_0 \otimes_R S \to M \otimes_R S \to 0.$$

Now, just observe that computing $\operatorname{Tor}_*^R(M, {}_R N)$ amounts to computing the homology of the chain

$$\ldots \to P_k \otimes_R N \to \ldots \to P_0 \otimes_R N \to 0$$

and that computing $\operatorname{Tor}_*^S(M \otimes_R S, N)$ amounts to computing the homology of

$$\ldots \to P_k \otimes_R S \otimes_S N \to \ldots \to P_0 \otimes_R S \otimes_S N \to 0$$

Since $S \otimes_S N \cong N$, the result follows. □

The second result mentioned above regarding Tor and universal localizations, combines Theorem 4.7 and 4.8 of [Sch85], and will be very useful in verifying condition (1) of Theorem 3.2.13 and Corollary 3.2.12.

Theorem 3.5.4. *Let $R \to S$ be an epic ring homomorphism. Then the following are equivalent:*

1. $\operatorname{Tor}_1^R(S, S) = 0$.
2. $\operatorname{Tor}_1^R(M, N) = \operatorname{Tor}_1^S(M, N)$ *for every right S-module M and every left S-module N.*
3. $\operatorname{Ext}_R^1(M, M') = \operatorname{Ext}_S^1(M, M')$ *for all right S-modules M and M'.*
4. $\operatorname{Ext}_R^1(N, N') = \operatorname{Ext}_S^1(N, N')$ *for all left S-modules N and N'.*

If $S = R_\Sigma$ is a universal localization of R, then all of these properties are satisfied.

The importance of this theorem is given by the fact that, since $\mathcal{D}_{\mathfrak{F}}$ is precisely the universal localization of $\mathfrak{F}$ with respect to the set of all full matrices, each of the statements in Theorem 3.5.4 holds for the epic embedding $\mathfrak{F} \hookrightarrow \mathcal{D}_{\mathfrak{F}}$, what will serve as the starting point for the proof of the main result.

We will now study the homological properties of the $\mathcal{S}$-module $\mathcal{D}_{\mathcal{S}}$ and its submodules. In particular, we will derive vanishing results for Tor and Ext, what will allow us to verify condition (1) and a weak version of condition (2) of Theorem 3.2.13 and Corollary 3.2.12. For this reason, the following is crucial in our setting.

Lemma 3.5.5. *Let R be a ring of right (resp. left) global dimension at most 1. Then any crossed product $R * \mathbb{Z}$ has right (resp. left) global dimension at most 2. In particular, if $\mathfrak{F}$ is a fir, then $\mathfrak{F} * \mathbb{Z}$ has right and left global dimension at most 2.*

Proof. By Proposition 3.4.10, $R * \mathbb{Z}$ can be seen as a skew Laurent polynomial $R[t^{\pm 1}; \tau]$ for some automorphism τ of R. Now [MR01, Theorem 7.5.3] applies (notice though some notational changes, since their polynomials are defined to be of the form $\sum_k (t^k a_k)$ to show that the right global dimension of $R * \mathbb{Z}$ is at most 2. The left version is obtained via an entirely symmetrical argument. The last statement follows because every fir $\mathfrak{F}$ has right and left global dimension at most 1. Indeed, if we take any left or right $\mathfrak{F}$-module M and any surjective homomorphism $\pi : F \to M$ where F is free, we have an exact sequence $0 \to \ker\pi \to F \xrightarrow{\pi} M \to 0$. Since $\mathfrak{F}$ is a fir and $\ker\pi$ is a submodule of the free module F, it must be free (see after Definition 1.1.7), and therefore $\operatorname{pd}(M) \leq 1$. □

We are now ready to study the homological properties of $\mathcal{D}_{\mathcal{S}}$ and its submodules.

Lemma 3.5.6.

(1) $\operatorname{Ext}^3_{\mathcal{S}}(M, M') = 0$ *for all left (resp. right) $\mathcal{S}$-modules M and M'.*

(2) $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ has projective dimension at most 1 as a left and right $\mathcal{S}$-module.

(3) Every left or right $\mathcal{S}$-submodule of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ has projective dimension at most 1.

(4) Every finitely generated left or right $\mathcal{S}$-submodule of $\mathcal{D}_{\mathcal{S}}$ has projective dimension at most 1.

Proof. (1) Since $\mathcal{S}$ has global dimension at most 2 by Lemma 3.5.5, this is a consequence of Lemma 3.2.7.

(2) Since $\mathfrak{F}$ has global dimension at most 1, the left $\mathfrak{F}$-module $\mathcal{D}_{\mathfrak{F}}$ admits a resolution $0 \to P_1 \to P_0 \to \mathcal{D}_{\mathfrak{F}} \to 0$ with P_1 and P_0 projective left $\mathfrak{F}$-modules. We now apply the functor $\mathcal{S} \otimes_{\mathfrak{F}} \square$ to this short exact sequence, where we view $\mathcal{S}$ as an $\mathcal{S}$-$\mathfrak{F}$-bimodule. Since $\mathcal{S}$ is a free right $\mathfrak{F}$-module, the resulting sequence is a projective resolution of the left $\mathcal{S}$-module $\mathcal{S} \otimes_{\mathfrak{F}} \mathcal{D}_{\mathfrak{F}}$, and thus the projective dimension of this module is at most 1. This finishes the proof, since the left $\mathcal{S}$-modules $\mathcal{S} \otimes_{\mathfrak{F}} \mathcal{D}_{\mathfrak{F}}$ and $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ are isomorphic by Lemma 3.4.7. The corresponding statement for the right $\mathcal{S}$-module $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ follows analogously.

(3) For every left (resp. right) $\mathcal{S}$-module M', the Ext long exact sequence obtained by applying the functor $\operatorname{Hom}_{\mathcal{S}}(\square, M')$ to the short exact sequence $0 \to M \to \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \to Q \to 0$ for an appropriate $\mathcal{S}$-module Q contains the following exact part:

$$\ldots \to \operatorname{Ext}^2_{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, M') \to \operatorname{Ext}^2_{\mathcal{S}}(M, M') \to \operatorname{Ext}^3_{\mathcal{S}}(Q, M') \to \ldots$$

Here, the first term vanishes by (2) and Lemma 3.2.7, and the third term vanishes by property (1). By exactness, we conclude that the term in the middle also vanishes. Thus, the claim follows from Lemma 3.2.7.

(4) This follows directly from (3) and Lemma 3.5.2. □

Lemma 3.5.7.

(1) $\operatorname{Tor}_1^{\mathfrak{F}}(\mathcal{D}_{\mathfrak{F}}, \mathcal{D}_{\mathfrak{F}}) = 0$.

(2) $\operatorname{Tor}_2^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) = 0$ *for every left $\mathcal{S}$-module N.*

(3) $\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) = 0$ *for every left $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$-module N.*

(4) $\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) = 0$ *for every left $\mathcal{S}$-submodule $N \leq \mathcal{D}_{\mathcal{S}}$.*

(5) $\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathcal{S}}, N) = 0$ *for every left $\mathcal{S}$-submodule $N \leq \mathcal{D}_{\mathcal{S}}$.*

(6) $\operatorname{Tor}_1^{\mathcal{S}}(N, \mathcal{D}_{\mathcal{S}}) = 0$ *for every right $\mathcal{S}$-submodule $N \leq \mathcal{D}_{\mathcal{S}}$.*

(7) $\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathcal{S}}, \mathcal{D}_{\mathcal{S}}) = 0$.

Proof. (1) Since $\mathfrak{F}$ is a fir, we know that $\mathcal{D}_{\mathfrak{F}}$ is the universal localization of $\mathfrak{F}$ with respect to the set of all full matrices, so this follows from Theorem 3.5.4.

(2) The flat dimension of a module is at most its projective dimension, so this follows from Lemma 3.5.6 (2) and Lemma 3.2.9.

(3) Observe that $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$ is isomorphic to $\mathcal{D}_{\mathfrak{F}} \otimes_{\mathfrak{F}} \mathcal{S}$ as a right $\mathcal{S}$-module by Lemma 3.4.7 and that $\mathcal{S}$ is a free left $\mathfrak{F}$-module (in particular flat). Thus, Lemma 3.5.3, together with (1) and Theorem 3.5.4 (2), tells us that

$$\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) \cong \operatorname{Tor}_1^{\mathfrak{F}}(\mathcal{D}_{\mathfrak{F}}, N) \cong \operatorname{Tor}_1^{\mathcal{D}_{\mathfrak{F}}}(\mathcal{D}_{\mathfrak{F}}, N) = 0.$$

(4) We have a short exact sequence $0 \to N \to \mathcal{D}_{\mathcal{S}} \to Q \to 0$ for some left $\mathcal{S}$-module Q. Applying $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} \square$ to this sequence, we obtain a long exact sequence that contains the following subsequence:

$$\ldots \to \operatorname{Tor}_2^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, Q) \to \operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) \to \operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, \mathcal{D}_{\mathcal{S}}) \to \ldots$$

Since the first and third term vanish by (2) and (3), respectively, we obtain the result.

(5) Let

$$\ldots \to P_k \to \ldots \to P_0 \to N \to 0$$

be a projective resolution of N. We can compute $\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathcal{S}}, N)$ as the first homology group of the $\mathcal{S}$-chain complex

$$\ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} P_k \to \ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} P_0 \to 0.$$

Since $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} \square \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} \square$, this complex is $\mathcal{S}$-isomorphic to:

$$C_* : \quad \ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_k \to \ldots \to \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_0 \to 0.$$

Using that $\mathcal{D}_{\mathcal{S}}$ is the Ore localization of $\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}$, which implies that the functor $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \square$ is exact, we obtain that $H_*(C_*) \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} H_*(D_*)$, where

$$D_* : \quad \ldots \to \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_k \to \ldots \to \mathcal{D}_{\mathfrak{F}} * \mathbb{Z} \otimes_{\mathcal{S}} P_0 \to 0.$$

But the homology of this complex computes $\operatorname{Tor}_k^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N)$, and thus

$$\operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathcal{S}}, N) \cong H_1(C_*) \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} H_1(D_*) \cong \mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}} \operatorname{Tor}_1^{\mathcal{S}}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z}, N) \overset{(4)}{=} 0.$$

(6) Every step in the proof of (5) can be adapted for right modules since $\mathcal{S}$ is also a free right $\mathfrak{F}$-module, and we can apply Lemma 3.5.6, Lemma 3.4.7 and the corresponding version of Lemma 3.5.3 for right modules.

(7) This is a special case of (5). □

We obtain from the previous results a weaker version of conditions (2) of Corollary 3.2.12 and Theorem 3.2.13:

Proposition 3.5.8. *For every finitely generated left or right $\mathcal{S}$-submodule M of $\mathcal{D}_{\mathcal{S}}$ and every exact sequence $0 \to J \to \mathcal{S}^n \to M \to 0$, the $\mathcal{S}$-module J is finitely generated projective.*

Proof. Since M has projective dimension at most 1 by Lemma 3.5.6 (4) and $\mathcal{S}^n$ is projective, we conclude from Lemma 3.2.7 that J is projective (in fact, this can be directly seen from Schanuel's lemma).

If M is a left $\mathcal{S}$-module and we apply the functor $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} \square$ to the short exact sequence defining J, the sequence remains exact by Lemma 3.5.7 (5). In particular, $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} J$ is isomorphic to a $\mathcal{D}_{\mathcal{S}}$-submodule of the finitely generated $\mathcal{D}_{\mathcal{S}}$-module $(\mathcal{D}_{\mathcal{S}})^n$. But $\mathcal{D}_{\mathcal{S}}$ is a division ring, thus $\mathcal{D}_{\mathcal{S}} \otimes_{\mathcal{S}} J$ is itself finitely generated. Since J is projective, [LLS03, Lemma 4] applies and we obtain that J is finitely generated. □

We finally have all the necessary ingredients for the proof of our main theorem for $\mathfrak{F} * \mathbb{Z}$.

Theorem 3.5.9. *Let $\mathfrak{F}$ be a fir with universal division $\mathfrak{F}$-ring of fractions $\mathcal{D}_{\mathfrak{F}}$, and consider a crossed product $\mathcal{S} = \mathfrak{F} * \mathbb{Z}$. Then, the following hold:*

a) $\mathcal{S}$ is a pseudo-Sylvester domain if and only if every finitely generated projective $\mathcal{S}$-module is stably free.

b) $\mathcal{S}$ is a Sylvester domain if and only if it is projective-free.

In any of the previous situations, $\mathcal{D}_{\mathcal{S}} = \mathcal{Q}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$ is the universal division $\mathcal{S}$-ring of fractions. Furthermore, it is isomorphic to the universal localization of $\mathcal{S}$ with respect to the set of all stably full (resp. full) matrices.

Proof. By Lemma 3.5.7 (7), the conditions (1) of Theorem 3.2.13 and Corollary 3.2.12 are satisfied for $\mathcal{S} \hookrightarrow \mathcal{D}_{\mathcal{S}}$, while we obtain from Proposition 3.5.8 that the module J appearing in the conditions (2) is finitely generated and projective. Therefore, if every finitely generated projective $\mathcal{S}$-module is stably free (resp. free), we deduce that $\mathcal{S}$ is a pseudo-Sylvester domain (resp. Sylvester domain). Conversely, over a pseudo-Sylvester domain every finitely generated projective module is stably free while Sylvester domains are projective-free (see Proposition 3.2.5).

In any of the previous cases, we conclude from the criteria that $\mathcal{D}_{\mathcal{S}} = \mathcal{Q}(\mathcal{D}_{\mathfrak{F}} * \mathbb{Z})$ is the universal division $\mathfrak{F} * \mathbb{Z}$-ring of fractions, and hence isomorphic to the localization of $\mathfrak{F} * \mathbb{Z}$ with respect to the set of all stably full (resp. full) matrices. □

As a particular application of Theorem 3.5.9 and the recent advances on the Farrell–Jones conjecture by Bestvina–Fujiwara–Wigglesworth and Brück–Kielak–Wu, we can obtain a stronger result for crossed products $E * G$ where E is a division ring and G arises as an extension $1 \to F \to G \to \mathbb{Z} \to 1$ with F a free group. By Proposition 3.4.11, $E * G$ can be expressed as an iterated crossed product $(E * F) * \mathbb{Z}$. Since $E * F$ is a fir (as we mentioned after the Definition 1.1.7), we are in the situation of Theorem 3.5.9 with $\mathfrak{F} = E * F$ and $\mathcal{S} = E * G$.

Note that A. Jaikin-Zapirain already showed in [Jai20B, Theorem 3.7] that in this case $E * G$ has a universal division ring of fractions. With Theorem 3.5.13 we provide an independent proof of this fact as well as a description of the matrices that become invertible over it. Furthermore, in [LL18, Theorem 2.17], it has already been shown that $K[G]$, where K is a subfield of $\mathbb{C}$, admits a universal localization that is a division ring.

The necessary tools for this theorem and some concrete examples are treated and developed separately in the next subsection.

3.5.1 The Farrell–Jones conjecture and stably freeness

As mentioned before, in this subsection we use recent results on the Farrell–Jones conjecture to improve Theorem 3.5.9 for the crossed products $E * G$ stated above. The following piece of the algebraic K-theory of a ring is needed to phrase the results:

Definition 3.5.10. Let R be a ring. Then we denote by $K_0(R)$ the abelian group generated by the isomorphism classes $[P]$ of finitely generated projective R-modules together with the relations

$$[P \oplus Q] - [P] - [Q] = 0$$

for all finitely generated projective R-modules P and Q.

Every element of $K_0(R)$ is of the form $[P] - [P']$ for finitely generated projective R-modules P and P'. The identity $[P] = [P'] \in K_0(R)$ holds for two finitely generated projective R-modules P and P' if and only if there is a finitely generated projective R-module Q such that $P \oplus Q \cong P' \oplus Q$, where Q can even be taken to be free.

If $\varphi : R \to S$ is a ring homomorphism and P is a finitely generated projective left R-module, then $S \otimes_R P$ is a finitely generated projective left S-module. In this way, $K_0(\square)$ becomes a functor from rings to abelian groups. Observe that, due to the relation between finitely generated projective left R-modules and finitely generated projective right R-modules described in the proof of Lemma 3.2.4, $K_0(R)$ does not depend on whether we use left or right projectives in its definition.

The *Farrell–Jones conjecture* makes far-reaching claims about the K-theory and L-theory of group rings or, more generally, additive categories with group actions, in particular for torsion-free groups. It is known for many classes of groups and satisfies a number of useful inheritance properties. For a full statement of the Farrell–Jones conjecture and an overview of the groups for which it is known, we refer the reader to the surveys [BLR08] and [RV18], and also to [Lüc10, Lüc19].

The following consequence of the Farrell–Jones conjecture, which is apparently well-known, has not been made explicit in the literature. The proof presented here is due to Fabian Henneke.

Proposition 3.5.11. *Let E be a division ring, Γ a torsion-free group and $E * \Gamma$ a crossed product. If the K-theoretic Farrell–Jones conjecture with coefficients in an additive category holds for Γ, then the embedding $E \hookrightarrow E * \Gamma$ induces an isomorphism*

$$K_0(E) \xrightarrow{\cong} K_0(E * \Gamma).$$

In particular, since $K_0(E) = \{n[E] \mid n \in \mathbb{Z}\}$, every finitely generated projective $E * \Gamma$-module is stably free.

Proof. For a given crossed product $E * \Gamma$, we will denote the additive category defined in [BR07, Corollary 6.17] by $\mathcal{A}_{E*\Gamma}$. We will freely use the terminology and notation of that paper. Furthermore, we will denote the family of virtually cyclic subgroups of a given group by VCyc and the family consisting just of the trivial subgroup by Tr. The K-theoretic Farrell–Jones conjecture for the group Γ with coefficients in the additive category $\mathcal{A}_{E*\Gamma}$ arises as an instance of the more general meta-isomorphism conjecture [Lüc19, Conjecture 13.2] for the Γ-homology theory $\mathcal{H}_*^{\Gamma}(\square; \mathbf{K}_{\mathcal{A}_{E*\Gamma}})$ introduced in [BR07] and the family $\mathcal{F} = \mathrm{VCyc}$. It states that the assembly map

$$\mathcal{H}_*^{\Gamma}(\mathcal{E}_{\mathrm{VCyc}}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \to \mathcal{H}_*^{\Gamma}(\mathrm{pt}; \mathbf{K}_{\mathcal{A}_{E*\Gamma}})$$

is an isomorphism, where the right-hand side is isomorphic to $K_*(E * \Gamma)$ by [BR07, Corollary 6.17].

In order to arrive at the desired conclusion, we need to reduce the family from VCyc to Tr. Since Γ is assumed to be torsion-free and hence all its virtually cyclic subgroups are infinite cyclic, we can arrange for this via the transitivity principle of [Lüc19, Theorem 13.13 (i)] if the meta-isomorphism conjecture with the $\mathbb{Z}$-homology theory $\mathcal{H}_*^{\mathbb{Z}}(\square; \mathbf{K}_{\mathcal{A}_{E*\mathbb{Z}}})$ and the family $\mathcal{F} = \mathrm{Tr}$ holds. A model for the classifying space $\mathcal{E}_{\mathrm{Tr}}(\mathbb{Z})$ is given by S^1 and we may again assume that the crossed product $E * \mathbb{Z}$ is a skew Laurent polynomial ring $E[t^{\pm 1}; \tau]$. In this situation, since E is regular (in the sense of [BL20, Definition 5.1]), the assembly map coincides with the map provided by the analogue of the Fundamental Theorem of algebraic K-theory for skew Laurent polynomial rings, which is an isomorphism (cf. [BL20, Theorems 6.8 & 9.1] or [Gra88] for a more classical treatment).

Since the K-theoretic Farrell–Jones conjecture with coefficients in an additive category is assumed to hold for G, we now obtain from the transitivity principle that the assembly map

$$\mathcal{H}_*^{\Gamma}(\mathcal{E}_{\mathrm{Tr}}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \to \mathcal{H}_*^{\Gamma}(\mathrm{pt}; \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \cong K_*(E * \Gamma)$$

is an isomorphism. The space $\mathcal{E}_{\mathrm{Tr}}$ is a free Γ-space and the value at the coset $\Gamma/\{1\}$ of the $\mathrm{Or}(\Gamma)$-spectrum $\mathbf{K}_{\mathcal{A}_{E*\Gamma}}$ is $\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})$. We can thus simplify the left-hand side of the assembly map as follows:

$$\mathcal{H}_*^{\Gamma}(\mathcal{E}_{\mathrm{Tr}}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \cong \mathcal{H}_*^{\Gamma}(\mathcal{E}(\Gamma); \mathbf{K}_{\mathcal{A}_{E*\Gamma}}) \cong H_*(B\Gamma; \mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})).$$

Here, $B\Gamma$ denotes the standard classifying space of the group Γ and homology is taken with local coefficients. Using [BR07, Corollary 6.17] once more, we note that $\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})$ is weakly equivalent to $\mathbb{K}^{-\infty}(E)$, which is connective by [Lüc19, Theorem 3.6] since E is a regular ring. In particular, the Atiyah–Hirzebruch spectral sequence provides the following natural isomorphism:

$$H_0(B\Gamma; \mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})) \cong H_0(B\Gamma; \pi_0(\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\}))),$$

where homology is again taken with local coefficients. Since $\pi_0(\mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})) \cong K_0(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})$ and the Γ-action on $\mathcal{A}_{E*\Gamma} * \Gamma/\{1\}$, which is induced from that on the Γ-space $\Gamma/\{1\}$, preserves isomorphism types, the local coefficients are in fact constant. We conclude that

$$H_0(B\Gamma; \mathbb{K}^{-\infty}(\mathcal{A}_{E*\Gamma} * \Gamma/\{1\})) \cong H_0(B\Gamma; K_0(E)),$$

and thus the assembly map in degree 0 simplifies to

$$K_0(E) \cong H_0(B\Gamma; K_0(E)) \xrightarrow{\cong} K_0(E * \Gamma).$$

This proves the first statement.

The second statement is now a consequence since every finitely generated projective $E * \Gamma$-module P represents an element $n[E * \Gamma]$ in $K_0(E * \Gamma)$ for $n \geq 0$, and thus there exists a finitely generated free $E * \Gamma$-module Q such that $P \oplus Q \cong (E * \Gamma)^n \oplus Q$, which is free. $\square$

The following is the K-theoretic part of [BFW19, Theorem 1.1] in the case of a finitely generated free group F and [BKW19, Theorem A] in the general case.

Theorem 3.5.12. *The K-theoretic Farrell–Jones conjecture with coefficients in an additive category holds for every group that arises as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

with F a (not necessarily finitely generated) free group.

We can finally state our main result for these crossed products. Here, we use $\mathcal{D}_{E*F}$ to denote the universal division $E * F$-ring of fractions.

Theorem 3.5.13. *Let E be a division ring and G a group arising as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is a free group. Then any crossed product $E * G$ is a pseudo-Sylvester domain, $\mathcal{D}_{E*G} = \mathcal{Q}(\mathcal{D}_{E*F} * \mathbb{Z})$ is the universal division $E*G$-ring of fractions and it is isomorphic to the universal localization of $E * G$ with respect to the set of all stably full matrices. Moreover, $E * G$ is a Sylvester domain if and only if it has stably free cancellation.

Proof. Since G satisfies the K-theoretic Farrell–Jones conjecture with coefficients in additive categories by Theorem 3.5.12, we obtain from Proposition 3.5.11 that every finitely generated projective $E * G$-module is stably free. Therefore, the statement follows from Theorem 3.5.9. $\square$

As a consequence of Proposition 3.4.26 and Theorem 3.4.27, we have the following realization of the universal division $E * G$ ring of fractions.

Corollary 3.5.14. *Under the hypothesis of Theorem 3.5.13, and if $\leq$ is any Conradian left order in G, then $\mathcal{D}_{E*G}$ can be realized as the division closure of $E * G$ inside* $\mathrm{End}(E((G, \leq)))$.

In the next chapter we give another realization of the universal division ring of fractions for such a G and a group ring $K[G]$ over a commutative field K, namely, the division closure of $K[G]$ inside $\mathcal{U}(G)$.

The main examples of groups of the form $1 \to F \to G \to \mathbb{Z} \to 1$ are the free-by-{infinite cyclic} groups and the fundamental groups of connected closed surfaces with genus $g \geq 1$ other than the projective plane (see Example 3.4.15(5) and (6)). Within these families, there are some cases of group rings for which it is known whether they admit stably free cancellation. In the following examples, K is any field of characteristic 0.

Example 3.5.15.

- Examples of group rings with stably free cancellation are $K[\mathbb{Z}^2] = K[S_1]$ and $K[F_2 \times \mathbb{Z}]$ (cf. [Bas64, Theorem 1] using that $K[\mathbb{Z}]$ is a PID; one can also consult [Swa78]) for the first example).

- Examples of group rings which do admit non-free stably free modules are given by $K[\mathbb{Z} \rtimes \mathbb{Z}] = K[\mathfrak{S}_2]$ (cf. [Sta85, Theorem 2.12]) and $\mathbb{Q}[\langle x, y \mid x^3 = y^2 \rangle] = \mathbb{Q}[F_2 \rtimes \mathbb{Z}]$ (cf. [Lew82] and note that the non-free projective ideal in the main theorem is actually stably free). Here, the latter example is the rational group ring of the fundamental group of the complement of the trefoil knot, which fibers over the circle and hence admits a free-by-{infinite cyclic} fundamental group (cf. [BZH13, Corollary 4.12]). Both group rings serve as examples of pseudo-Sylvester domains that are not Sylvester domains.

□

At the time of writing [HL20], it was an open question (to the best of the authors' knowledge) whether $\mathbb{C}[S_g]$ for $g \geq 2$ and $\mathbb{C}[\mathfrak{S}_g]$ for $g \geq 3$ have stably free cancellation.

Chapter 4

The strong Atiyah conjecture for locally indicable groups

This chapter is based on [JL20, Sections 2 to 6], and it is devoted to present the strong Atiyah conjecture, which in the form presented here is usually attributed to W. Lück and T. Schick, and the main ingredients needed in order to show that it holds for the family of locally indicable groups.

Let G be a group and let K be a subfield of $\mathbb{C}$. Our motivation to the study of the strong Atiyah conjecture for locally indicable groups comes from the fact that, when the group G is torsion-free, the conjecture is deeply related to the question of embeddability of $K[G]$ into a division ring. More precisely, there exists a regular ring $\mathcal{U}(G)$, together with a faithful Sylvester matrix rank function rk_G, in which $K[G]$ embeds, and rk_G takes integer values on matrices over $K[G]$ if and only if the division closure $\mathcal{D}_{K[G]}$ of $K[G]$ inside $\mathcal{U}(G)$ is a division ring. In addition, when G is locally indicable, $\mathcal{D}_{K[G]}$ is the natural candidate to be the Hughes-free division ring of fractions for $K[G]$.

The aforementioned ring $\mathcal{U}(G)$ is not only regular but comes equipped with a proper involution $*$, i.e., it is a $*$-regular ring. Although at first sight it may look like the conditions of being regular and having a proper involution do not interact, it turns out that $*$-regular rings enjoy stronger properties than regular rings. In particular, in a $*$-regular ring $\mathcal{U}$, every finitely generated left or right ideal is generated by a unique projection, what a posteriori implies the existence of the smallest $*$-regular subring of $\mathcal{U}$ containing a given $*$-subring ([AG17, Proposition 6.2]). We use this, together with the theory of epic $*$-regular rings developed by A. Jaikin-Zapirain in [Jai19], which is parallel to the theory of epic division rings, to tackle the strong Atiyah conjecture.

The last tool for the proof of the main results is the notion of complexity developed in [DHS04]. Given a subring R of a ring S, we can assign a complexity to each element x of the division closure $\mathcal{D}_{R,S}$, which somehow measures how "deep" in the inductive construction of $\mathcal{D}_{R,S}$ we need to go in order to find x. This, together with the defining property of locally indicable groups, allows us to make proofs by induction on the complexity of elements.

The chapter is organized as follows. We recall the basic theory of $*$-regular and epic

$*$-regular rings in Section 4.1, while Section 4.2 is devoted to introduce and describe $\mathcal{U}(G)$, rk_G, and the strong Atiyah conjecture for $K[G]$. In Section 4.3 we define the notion of complexity used to prove finally in Section 4.4 the strong Atiyah conjecture for locally indicable groups and its direct consequences.

4.1 $*$-regular and epic $*$-regular rings

In this section we recall some of the basic properties of $*$-regular rings and introduce the theory of epic $*$-regular rings, which is parallel to Cohn's theory of epic division rings presented in Section 3.1.

Definition 4.1.1. Let R be a ring. An *involution* is a map $*: R \to R$ that satisfies the following properties for all $x, y \in R$.

(1) $(x+y)^* = x^* + y^*$.

(2) $(xy)^* = y^* x^*$.

(3) $(x^*)^* = x$.

(4) $(1_R)^* = 1_R$.

A ring with involution is called a $*$-*ring*. We say that the involution $*$ is *proper* if it additionally satisfies

(5) $x^*x = 0$ implies $x = 0$.

A $*$-*regular* ring is a (von Neumann) regular ring with a proper involution.

One of the most important features of a $*$-regular ring $\mathcal{U}$ is that every finitely generated ideal is generated by a unique projection, i.e., by a self-adjoint ($e^* = e$) and idempotent ($e^2 = e$) element $e \in \mathcal{U}$. As we show in the next proposition, which has been extracted from [Jai19, Proposition 3.2], this allows us to distinguish, for every $x \in \mathcal{U}$, a particular element $y \in \mathcal{U}$ with the property that $xyx = x$.

Proposition 4.1.2. *Let $\mathcal{U}$ be a $*$-regular ring and let x be an element of $\mathcal{U}$. The following hold.*

(i) $\mathcal{U}x = \mathcal{U}x^*x$ *and* $x\mathcal{U} = xx^*\mathcal{U}$.

(ii) *There exist unique projections e and f such that $\mathcal{U}x = \mathcal{U}e$ and $x\mathcal{U} = f\mathcal{U}$. We write* $f = \mathrm{LP}(x)$ *and* $e = \mathrm{RP}(x)$.

(iii) $\mathrm{RP}(x) = \mathrm{RP}(x^*x) = \mathrm{LP}(x^*)$ *and* $\mathrm{LP}(x) = \mathrm{LP}(xx^*) = \mathrm{RP}(x^*)$.

(iv) *There exist a unique element $y \in e\mathcal{U}f$ such that $yx = e$. Moreover, $xy = f$, $xyx = x$ and $yxy = y$. We write $y = x^{[-1]}$ and we call it the* relative inverse of x. *Furthermore, $y^{[-1]} = x$.*

(*v*) $(x^{[-1]})^* = (x^*)^{[-1]}$, $(x^*x)^{[-1]} = x^{[-1]}(x^*)^{[-1]}$ *and* $x^{[-1]} = (x^*x)^{[-1]}x^*$.

(*vi*) *If x is self-adjoint, then x and $x^{[-1]}$ commute.*

Proof.
(*i*) Let $z \in \mathcal{U}$ be such that $x^*xzx^*x = x^*x$, and write $t = zx^*x - 1$. Then $x^*xt = 0$, from where $0 = t^*x^*xt = (xt)^*xt$. Since $*$ is proper, we have $xt = 0$, i.e., $xzx^*x = x$, and therefore $\mathcal{U}x \subseteq \mathcal{U}x^*x$. Since we always have the other containment, this implies $\mathcal{U}x = \mathcal{U}x^*x$. Applying $*$ we see that $x^*\mathcal{U} = x^*x\mathcal{U}$, and since this is true for every x, we also have $x\mathcal{U} = xx^*\mathcal{U}$.

(*ii*) Set $f = xzx^*$. Then $f\mathcal{U} \subseteq x\mathcal{U}$, and from the relation $xzx^*x = x$, we deduce that $fx = x$, so that $x\mathcal{U} = f\mathcal{U}$. Moreover,

$$f = xzx^* = xz(xzx^*x)^* = (xzx^*)(xz^*x^*) = ff^*,$$

so $f = f^*$ and consequently $f = f^2$, i.e., f is a projection. If f' is another projection such that $f\mathcal{U} = f'\mathcal{U}$, then $f = f'f = (ff')^* = (f')^* = f'$. We can proceed analogously to prove the existence and uniqueness of e.

(*iii*) The first equality comes from (*i*), while applying $*$ to the equality $\mathcal{U}\,\mathrm{RP}(x) = \mathcal{U}x$ we obtain that $x^*\mathcal{U} = \mathrm{RP}(x)\mathcal{U}$. By the uniqueness in (*ii*), it must necessarily be the case that $\mathrm{RP}(x) = \mathrm{LP}(x^*)$. The second part is proved similarly.

(*iv*) Since $\mathcal{U}x = \mathcal{U}e$, there exists an element $y \in \mathcal{U}$ such that $yx = e$, and therefore $eyx = e$. Moreover, from $x\mathcal{U} = f\mathcal{U}$, we have that $fx = x$, and hence $e = eyx = (eyf)x$. Thus, substituting y by eyf we can assume that $y \in e\mathcal{U}f$ and $yx = e$. If $y' \in e\mathcal{U}f$ is another element satisfying $y'x = e$, then $(y - y')x = 0$, and since $f \in x\mathcal{U}$, we have $(y - y')f = 0$. But $yf = y$ and $y'f = y'$, so $y = y'$.

Furthermore, $xyx = xe = x$, $yxy = ey = y$, and proceeding as before, from $(xy - f)x = xyx - fx = xe - fx = 0$ we obtain that $0 = (xy - f)f = xyf - f^2 = xy - f$, so $xy = f$. Finally, the latter equality together with the fact that $y = yf$ implies that $\mathcal{U}y = \mathcal{U}f$, i.e., $f = \mathrm{RP}(y)$, and similarly $e = \mathrm{LP}(y)$. Therefore, since $xy = f$, the uniqueness of the relative inverse implies that $y^{[-1]} = x$.

(*v*) From (*iii*), $\mathrm{RP}(x^*) = \mathrm{LP}(x) = f$ and $\mathrm{LP}(x^*) = \mathrm{RP}(x) = e$, and by (*iv*) we have $xx^{[-1]} = f$. Thus, $(x^{[-1]})^* \in f\mathcal{U}e$ and $(x^{[-1]})^*x^* = f$, from where the uniqueness of the relative inverse implies $(x^{[-1]})^* = (x^*)^{[-1]}$.

Similarly $\mathrm{RP}(x^*x) = \mathrm{RP}(x) = e$ and since x^*x is self-adjoint, we also deduce that $\mathrm{LP}(x^*x) = e$. By the previous reasoning, $x^{[-1]}(x^*)^{[-1]} \in e\mathcal{U}e$ and

$$x^{[-1]}(x^*)^{[-1]}x^*x = x^{[-1]}fx = x^{[-1]}x = e,$$

what means that $(x^*x)^{[-1]} = x^{[-1]}(x^*)^{[-1]}$. Multiplying this expression by x^* we finally obtain that

$$(x^*x)^{[-1]}x^* = x^{[-1]}(x^*)^{[-1]}x^* = x^{[-1]}f = x^{[-1]}.$$

(*vi*) Since $xx^{[-1]} = (x^{[-1]})^*x^* = f$ and, from (*v*), $(x^{[-1]})^* = (x^*)^{[-1]}$, we obtain if x is self-adjoint that $xx^{[-1]} = x^{[-1]}x$. □

The following remark can be sometimes useful to identify the relative inverse or the projections associated to an element.

Remark 4.1.3. If $\mathcal{U}$ is a $*$-regular ring, the relative inverse of an element $x \in \mathcal{U}$ can also be defined as the unique element $y \in \mathcal{U}$ such that both xy and yx are projections and $xyx = x$, $yxy = y$. Indeed, from Proposition 4.1.2(iv) we see that the relative inverse satisfies these properties. Conversely, from $xyx = x$ we can see that $\mathcal{U}yx = \mathcal{U}x$ and $xy\mathcal{U} = x\mathcal{U}$, and hence we deduce by the uniqueness in Proposition 4.1.2(ii) that $yx = \mathrm{RP}(x) = e$, $xy = \mathrm{LP}(x) = f$. Finally, from $yxy = y$ we have

$$y = yxy = (yx)y(xy) = eyf \in e\mathcal{U}f,$$

and consequently $y = x^{[-1]}$.

On the other hand, if $e \in \mathcal{U}$ is a projection, then $e = \mathrm{RP}(x)$ if and only if $\mathrm{ann}_r(x) = (1 - e)\mathcal{U}$, where $\mathrm{ann}_r(x)$ denotes the right annihilator of x in $\mathcal{U}$. Indeed, from $\mathcal{U}x = \mathcal{U}\,\mathrm{RP}(x)$ one obtains $\mathrm{ann}_r(x) = \mathrm{ann}_r(\mathrm{RP}(x)) = (1 - \mathrm{RP}(x))\mathcal{U}$. This proves the direct implication and also that, if e is such that $\mathrm{ann}_r(x) = (1 - e)\mathcal{U}$, then $(1 - \mathrm{RP}(x))\mathcal{U} = (1 - e)\mathcal{U}$, from where the uniqueness of the projection implies $\mathrm{RP}(x) = e$. □

A direct consequence of the results in Proposition 4.1.2 is that a quotient of a $*$-regular ring is again $*$-regular (cf. [Jai19, Proposition 3.3]).

Corollary 4.1.4. *Let $\mathcal{U}$ be a $*$-regular ring and let I be a two-sided ideal of $\mathcal{U}$. Then I is $*$-closed and $*$ induces a proper involution in $\mathcal{U}/I$, i.e., $\mathcal{U}/I$ is a $*$-regular ring.*

Proof. Take $x \in I$. Then, by Proposition 4.1.2(i), $x^* \in \mathcal{U}xx^* \subseteq I$ and hence I is $*$-closed. In particular, if $x + I = y + I$, then $x^* - y^* = (x - y)^* \in I$, and hence $(x + I)^* := x^* + I$ defines an involution in $\mathcal{U}/I$. Moreover, again by Proposition 4.1.2(i), if $x^*x \in I$ then $x \in \mathcal{U}x^*x \subseteq I$, and therefore the involution in $\mathcal{U}/I$ is proper. Since $\mathcal{U}/I$ is regular, this finishes the proof. □

In a regular ring, the set of all principal left (right) ideals partially ordered by inclusion forms a (complemented modular) lattice with the sum and intersection of ideals as lattice operations (see [Goo91, Theorem 2.3]). In a $*$-regular ring $\mathcal{U}$, the set of projections can also be given a lattice structure in such a way that it is order-isomorphic to the previous one. Let us record this in the following proposition, which corresponds to [Goo91, Theorem 2.3]and [Ber72, Chapter 1, §3, Proposition 7].

Proposition 4.1.5. *Let $\mathcal{U}$ be a $*$-regular ring. Let $\mathcal{I}$ be the set of principal left ideals of $\mathcal{U}$, partially ordered by inclusion, and let $\mathcal{P}$ be the set of all projections in $\mathcal{U}$, partially ordered by setting $e \leq f$ if and only if $ef = e$. Then $\mathcal{I}$ and $\mathcal{P}$ form lattices with the operations given by*

($\mathcal{I}$). If $I, J \in \mathcal{I}$, then $I \wedge J = I \cap J$ and $I \vee J = I + J$.

($\mathcal{P}$). If $e, f \in \mathcal{P}$, then $e \wedge f = e - \mathrm{LP}(e(1 - f))$ and $e \vee f = f + \mathrm{RP}(e(1 - f))$.

Moreover, the map $\mathcal{P} \to \mathcal{I}$ given by $e \to \mathcal{U}e$ is an order-isomorphism.

Proof. As we mentioned above, that $\mathcal{I}$ forms a lattice with the defined operations is true in general for regular rings, as can be seen in [Goo91, Theorem 2.3] for the case $A = {}_{\mathcal{U}}\mathcal{U}$ (Recall that in a regular ring every finitely generated ideal is principal). Let us see the result for $\mathcal{P}$.

On the one hand, set $p = \mathrm{LP}(e(1-f))$, i.e., p is the unique projection such that $e(1-f)\mathcal{U} = p\mathcal{U}$, and hence $ep = p$. Since e and p are projections, applying $*$ we also deduce $pe = p$, and hence we see that $e-p$ is self-adjoint and satisfies

$$(e-p)^2 = e^2 - ep - pe + p^2 = e - p - p + p = e - p,$$

i.e., $e-p \in \mathcal{P}$. Moreover, from $e(1-f)\mathcal{U} = p\mathcal{U}$ we also have $e(1-f) = pe(1-f) = p(1-f)$, and therefore $(e-p)(1-f) = 0$. Thus,

$$e - p = (e-p)f + (e-p)(1-f) = (e-p)f$$

and

$$(e-p)e = e^2 - pe = e - p,$$

i.e. $e-p \leq f$ and $e-p \leq e$. Finally, if p' is another projection with $p' \leq f$ and $p' \leq e$, then $p' = p'e = p'f$. From here, $p'e(1-f) = p'(1-f) = p' - p'f = 0$, what implies that $p'p = 0$ by definition of p. Therefore, $p'(e-p) = p'$ and hence $p' \leq e-p$. This finishes the proof of $e \wedge f = e - \mathrm{LP}(e(1-f))$.

On the other hand, set $q = \mathrm{RP}(e(1-f))$, i.e., q is the unique projection such that $\mathcal{U}e(1-f) = \mathcal{U}q$, and hence $qf = 0$. As before, this also implies that $fq = 0$ and therefore the self-adjoint element $f+q$ is also idempotent, i.e., $f+q \in \mathcal{P}$. In addition, from $\mathcal{U}e(1-f) = \mathcal{U}q$ we also obtain $e(1-f) = e(1-f)q = eq$, and hence $e = e(f+q)$. Since we also have $f(f+q) = f$, this implies that $e \leq f+q$ and $f \leq f+q$. If q' is another projection with $e \leq q'$ and $f \leq q'$, then $eq' = e$ and $fq' = f$. Thus,

$$e(1-f)q' = e(q'-f) = e - ef = e(1-f),$$

from where the definition of q implies that $qq' = q$. As a consequence, $(f+q)q' = f+q$ and $f+q \leq q'$, what means that $e \vee f = f + \mathrm{RP}(e(1-f))$.

We have seen so far that $\mathcal{P}$ forms a lattice with the given operations, and now it follows from the existence and uniqueness in Proposition 4.1.2(ii) that the map $\mathcal{P} \to \mathcal{I}$ given by $e \to \mathcal{U}e$ is bijective. To finish, note that the map is an order-isomorphism, because $e \leq f$ if and only if $ef = e$ if and only if $\mathcal{U}e \subseteq \mathcal{U}f$. $\square$

As a corollary of the previous result, if we know the projections that generate the finitely generated left ideals I and J of the $*$-regular ring $\mathcal{U}$, then we can identify the generator of $I \cap J$ and $I + J$.

Corollary 4.1.6. *Let $\mathcal{U}$ be a $*$-regular ring and let $e, f \in \mathcal{U}$ be projections. Then*

$$\mathcal{U}e \cap \mathcal{U}f = \mathcal{U}(e \wedge f) \quad \text{and} \quad \mathcal{U}e + \mathcal{U}f = \mathcal{U}(e \vee f)$$

Proof. Since by Proposition 4.1.5, the map $\mathcal{P} \to \mathcal{I}$ sending the projection e to the left ideal $\mathcal{U}e$ is an order-isomorphism and both $\mathcal{P}$ and $\mathcal{I}$ are lattices, the element $e \wedge f$ must be sent to $\mathcal{U}e \wedge \mathcal{U}f = \mathcal{U}e \cap \mathcal{U}f$, i.e., $\mathcal{U}e \cap \mathcal{U}f = \mathcal{U}(e \wedge f)$. Similarly $\mathcal{U}e + \mathcal{U}f = \mathcal{U}(e \vee f)$. □

Remark 4.1.7. Observe that the same result holds for right ideals. Indeed, let e and f be projections in $\mathcal{U}$. From the defining properties (1) and (2) of an involution, we see that $(\mathcal{U}e + \mathcal{U}f)^* = (\mathcal{U}e)^* + (\mathcal{U}f)^* = e\mathcal{U} + f\mathcal{U}$. Similarly, from the defining properties (3) and (2) of an involution, we can see that $(\mathcal{U}e \cap \mathcal{U}f)^* = (\mathcal{U}e)^* \cap (\mathcal{U}f)^* = e\mathcal{U} \cap f\mathcal{U}$. Therefore, since $e \wedge f$ and $e \vee f$ are projections, we deduce applying $*$ to the expressions in the corollary that

$$e\mathcal{U} \cap f\mathcal{U} = (e \wedge f)\mathcal{U} \quad \text{and} \quad e\mathcal{U} + f\mathcal{U} = (e \vee f)\mathcal{U}$$

□

Now, let us develope the theory of epic $*$-regular rings. Recall from Chapter 3 that, given a ring R, an epic division R-ring is a division ring $\mathcal{D}$ together with a ring homomorphism $\varphi : R \to \mathcal{D}$ such that $\mathcal{D}$ is generated as a division ring by $\varphi(R)$, or in other words, such that $\mathcal{D} = \mathcal{D}_{\varphi(R),\mathcal{D}}$. If R is a $*$-subring of a $*$-regular ring $\mathcal{U}$, P. Ara and K. R. Goodearl realized that, in a way similar to that of the division closure, one can construct the smallest $*$-regular subring of $\mathcal{U}$ containing R. The content of the following proposition corresponds to [AG17, Proposition 6.2].

Proposition 4.1.8. *Let R be a $*$-subring of a $*$-regular ring $\mathcal{U}$. Then there exists a smallest $*$-regular subring $\mathcal{R}(R,\mathcal{U})$ of $\mathcal{U}$ containing R. Moreover, it can be constructed as follows.*

1. *Set $\mathcal{R}_0(R,\mathcal{U}) := R$, a $*$-subring of $\mathcal{U}$.*

2. *Suppose $n \geq 0$ and that we have constructed a $*$-subring $\mathcal{R}_n(R,\mathcal{U})$ of $\mathcal{U}$. Let $\mathcal{R}_{n+1}(R,\mathcal{U})$ be the subring of $\mathcal{U}$ generated by the elements of $\mathcal{R}_n(R,\mathcal{U})$ and their relative inverses in $\mathcal{U}$, which is a $*$-subring.*

Then $\mathcal{R}(R,\mathcal{U}) = \bigcup_{n=0}^{\infty} \mathcal{R}_n(R,\mathcal{U})$.

Proof. Let $\{\mathcal{U}_i\}_{i\in I}$ be the family of all $*$-regular subrings of $\mathcal{U}$ that contain R, and set $\mathcal{R}(R,\mathcal{U}) = \bigcap_{i\in I} \mathcal{U}_i$. Observe that $\mathcal{R}(R,\mathcal{U})$ is a $*$-subring of $\mathcal{U}$. In order to show that $\mathcal{R}(R,\mathcal{U})$ is regular, take $x \in \mathcal{R}(R,\mathcal{U})$ and note by Remark 4.1.3 that the relative inverse of x in $\mathcal{U}_i$ must coincide with the relative inverse $x^{[-1]}$ of x in $\mathcal{U}$. Therefore, $x^{[-1]} \in \mathcal{U}_i$ for all i and hence $x^{[-1]} \in \mathcal{R}(R,\mathcal{U})$. Thus, $\mathcal{R}(R,\mathcal{U})$ is regular, and since $*$ is proper, it is a $*$-regular subring of $\mathcal{U}$ containing R, clearly the smallest one.

Now, assume that S is a $*$-subring of $\mathcal{U}$, and consider the subring S' of $\mathcal{U}$ generated by the elements of S and their relative inverses in $\mathcal{U}$. If $x \in S$, then $x^* \in S \subseteq S'$ by hypothesis, and if $y = x^{[-1]}$ for some $x \in S$, then Proposition 4.1.2(v) tells us that $y^* = (x^{[-1]})^* = (x^*)^{[-1]}$, from where since $x^* \in S$ we deduce that $y^* \in S'$ by definition. This implies that the adjoints of the generators of S' live in S', and hence by the properties of an involution, this is enough to show that S' is a $*$-subring.

In particular, each $\mathcal{R}_n(R,\mathcal{U})$ is a $*$-subring of $\mathcal{U}$, and since $\mathcal{R}_n(R,\mathcal{U}) \subseteq \mathcal{R}_{n+1}(R,\mathcal{U})$, we have that $S = \bigcup_{n=0}^{\infty} \mathcal{R}_n(R,\mathcal{U})$ is a $*$-subring of $\mathcal{U}$ containing R. Moreover, it is regular by construction since the relative inverse of an element $x \in \mathcal{R}_n(R,\mathcal{U})$ lies in $\mathcal{R}_{n+1}(R,\mathcal{U}) \subseteq S$. One can inductively show that for every $i \in I$ and every n, $\mathcal{R}_n(R,\mathcal{U}) \subseteq \mathcal{U}_i$, and therefore $S = \mathcal{R}(R,\mathcal{U})$. □

Definition 4.1.9. Let R be a $*$-subring of a $*$-regular ring $\mathcal{U}$. Then $\mathcal{R}(R,\mathcal{U})$ is called the *$*$-regular closure of R in $\mathcal{U}$*.

The next lemma is some sort of analog of Lemma 3.3.3(2) for the $*$-regular closure.

Lemma 4.1.10. *Let $R \subseteq \mathcal{U} \subseteq \mathcal{U}'$ be a chain of proper $*$-(sub)rings such that $\mathcal{U}$ and $\mathcal{U}'$ are $*$-regular. Then $\mathcal{R}(R,\mathcal{U})$ is division closed and $\mathcal{R}(R,\mathcal{U}) = \mathcal{R}(R,\mathcal{U}')$.*

Proof. The first claim follows because if $x \in \mathcal{R}(R,\mathcal{U})$ is invertible in $\mathcal{U}$, then by uniqueness of the relative inverse we have $x^{-1} = x^{[-1]} \in \mathcal{R}(R,\mathcal{U})$. For the second claim, since $\mathcal{R}(R,\mathcal{U})$ is a $*$-regular subring of $\mathcal{U} \subseteq \mathcal{U}'$ containing R, we have by definition that $\mathcal{R}(R,\mathcal{U}') \subseteq \mathcal{R}(R,\mathcal{U}) \subseteq \mathcal{U}$, but this implies that $\mathcal{R}(R,\mathcal{U}')$ is actually a $*$-regular subring of $\mathcal{U}$ containing R, from where we obtain the other containment, and hence the equality. □

The $*$-regular closure will play the role of the division closure in the theory of epic $*$-regular rings, namely, given a $*$-ring R we are going to work with $*$-regular rings $\mathcal{U}$ and $*$-homomorphisms $R \to \mathcal{U}$ such that $\mathcal{U} = \mathcal{R}(R,\mathcal{U})$. As it happens with epic division rings (see Proposition 3.1.13), the adjective "epic" is not a coincidence. To show that, we need the following characterization of epicity, which can be found for instance in [Coh06, Proposition 7.2.1].

Proposition 4.1.11. *Let $\varphi : R \to S$ be a ring homomorphism. The following are equivalent.*

1. *φ is epic.*
2. *In the S-bimodule $S \otimes_R S$, we have $s \otimes 1_S = 1_S \otimes s$ for all $s \in S$.*
3. *The multiplication map $m : S \otimes_R S \to S$ given by $s \otimes s' \mapsto ss'$ is an isomorphism of S-bimodules.*

As a first application of this proposition, let us show the following result regarding epic homomorphisms and centers of rings.

Corollary 4.1.12. *If R is a subring of S and the embedding $R \hookrightarrow S$ is epic, then $Z(R) \subseteq Z(S)$.*

Proof. For every $a \in Z(R)$, the map $S \times S \to S \otimes_R S$ given by $(s, s') \mapsto s \otimes as'$ is R-biadditive, and hence we have a well-defined map $\phi : S \otimes_R S \to S \otimes_R S$ with $\phi(s \otimes s') = s \otimes as'$. Since the embedding of R in S is epic, the previous proposition tells us that for every $s \in S$, the equality $s \otimes 1_S = 1_S \otimes s$ holds in $S \otimes_R S$, and hence $s \otimes a = \phi(s \otimes 1_S) = \phi(1_S \otimes s) = 1_S \otimes as$. Applying the multiplication map m, we have $sa = as$ in S. □

In the same way that, when "measuring" the surjectivity of a ring homomorphism we consider its image, i.e., the set of elements in the codomain with a preimage, when "measuring" epicity we can consider its dominion, i.e., the set of elements of the codomain that satisfy the epicity property. More precisely,

Definition 4.1.13. Let $\varphi : R \to S$ be a ring homomorphism. The *dominion of* φ is the subset $\mathrm{dmn}(\varphi)$ of S consisting of the elements $s \in S$ such that, for every pair of ring homomorphisms $\psi_1, \psi_2 : S \to Q$ satisfying $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ we have $\psi_1(s) = \psi_2(s)$.

Note that $\mathrm{dmn}(\varphi)$ is actually a subring of S and that φ is epic if and only if $\mathrm{dmn}(\varphi) = S$. Assume that we have ring homomorphisms $\varphi_1 : R_1 \to R_2$ and $\varphi_2 : R_2 \to R_3$. We know that if φ_1 is surjective, then $\mathrm{im}(\varphi_2 \circ \varphi_1) = \mathrm{im}\,\varphi_2$. Analogously, if φ_1 is epic, then

$$\mathrm{dmn}(\varphi_2 \circ \varphi_1) = \mathrm{dmn}(\varphi_2).$$

Indeed, note that since φ_1 is epic, given two ring homomorphisms $\psi_1, \psi_2 : R_3 \to Q$, we have $\psi_1 \circ \varphi_2 = \psi_2 \circ \varphi_2$ if and only if $\psi_1 \circ \varphi_2 \circ \varphi_1 = \psi_2 \circ \varphi_2 \circ \varphi_1$, and hence an element s is in $\mathrm{dmn}(\varphi_2 \circ \varphi_1)$ if and only if it belongs to $\mathrm{dmn}(\varphi_2)$.

A feature of regular rings is that, for a ring homomorphism whose domain is regular, the dominion and the image coincide, and hence in particular any epic homomorphism from a regular ring is surjective. Let us record this in the following proposition (see [Ste75, Chapter XI, Proposition 1.4]).

Proposition 4.1.14. *Let $\mathcal{U}$ be a regular ring and let $\varphi : \mathcal{U} \to S$ be a ring homomorphism. Then $\mathrm{dmn}(\varphi) = \mathrm{im}(\varphi)$. In particular, φ is epic if and only if it is surjective.*

Proof. Factor φ through $\mathcal{U} \xrightarrow{\varphi} \varphi(\mathcal{U}) \overset{i}{\hookrightarrow} S$. The previous discussion tells us that $\mathrm{dmn}(i \circ \varphi) = \mathrm{dmn}(i)$, while since $\varphi(\mathcal{U})$ is regular, [Ste75, Chapter XI, Proposition 1.4] tells us that $\mathrm{dmn}(i) = \varphi(\mathcal{U})$. Therefore $\mathrm{dmn}(\varphi) = \mathrm{im}(\varphi)$. $\square$

The next corollary (cf. [Jai19, Corollary 4.3]) will later help us to prove the analog of Corollary 3.1.15 for epic $*$-regular rings.

Corollary 4.1.15. *Assume that we have a commutative diagram of rings and ring homomorphisms*

$$\begin{tikzcd} R \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathcal{U}_1 \arrow[d, "\gamma 1"] \\ \mathcal{U}_2 \arrow[r, "\gamma_2"'] & S \end{tikzcd}$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are regular and f_1 and f_2 are epic. Then $\gamma_1(\mathcal{U}_1) = \gamma_2(\mathcal{U}_2)$.

Proof. By the previous proposition, and since f_1 and f_2 are epic (see the discussion above),

$$\gamma_1(\mathcal{U}_1) = \mathrm{dmn}(\gamma_1) = \mathrm{dmn}(\gamma_1 \circ f_1) = \mathrm{dmn}(\gamma_2 \circ f_2) = \mathrm{dmn}(\gamma_2) = \gamma_2(\mathcal{U}_2).$$

$\square$

Moreover, with the previous proposition we can finally prove that, in analogy to Proposition 3.1.13 for a $*$-homomorphism $\varphi : R \to \mathcal{U}$ (i.e., a ring homomorphism that preserves the involution) with $\mathcal{U}$ $*$-regular, the property that φ is epic is equivalent to the condition $\mathcal{U} = \mathcal{R}(\varphi(R), \mathcal{U})$. The proof has been extracted from [Jai19, Proposition 6.1]).

Proposition 4.1.16. *Let $\mathcal{U}$ be a $*$-regular ring and let $\varphi : R \to \mathcal{U}$ be a $*$-homomorphism. Then φ is epic if and only if $\mathcal{U} = \mathcal{R}(\varphi(R), \mathcal{U})$.*

Proof. Assume that $\mathcal{U} = \mathcal{R}(\varphi(R), \mathcal{U})$, and consider the set

$$S = \{s \in \mathcal{U} : s \otimes 1_{\mathcal{U}} = 1_{\mathcal{U}} \otimes s \text{ in } \mathcal{U} \otimes_R \mathcal{U}\}.$$

S clearly contains $\varphi(R)$ and moreover, if $s, s' \in S$, then on the one hand,

$$(s - s') \otimes 1_{\mathcal{U}} = s \otimes 1_{\mathcal{U}} - s' \otimes 1_{\mathcal{U}} = 1_{\mathcal{U}} \otimes s - 1_{\mathcal{U}} \otimes s' = 1_{\mathcal{U}} \otimes (s - s'),$$

and, on the other hand, using the $\mathcal{U}$-bimodule structure,

$$\begin{aligned} ss' \otimes 1_{\mathcal{U}} &= s(s' \otimes 1_{\mathcal{U}}) = s(1_{\mathcal{U}} \otimes s') = s \otimes s' \\ &= (s \otimes 1_{\mathcal{U}})s' = (1_{\mathcal{U}} \otimes s)s' = 1_{\mathcal{U}} \otimes ss'. \end{aligned}$$

Therefore $s - s', ss' \in S$ and S is a subring of $\mathcal{U}$.

Now, assume that $s \in S$ is self-adjoint and set $x = s^{[-1]} \in \mathcal{U}$. In particular, Proposition 4.1.2(iv) and (vi) tell us that $xs = sx$, $xsx = x$ and $sxs = s$. Thus, we have

$$\begin{aligned} x \otimes 1_{\mathcal{U}} &= xsx \otimes 1_{\mathcal{U}} = xx(s \otimes 1_{\mathcal{U}}) = xx(1_{\mathcal{U}} \otimes s) = xx(1_{\mathcal{U}} \otimes sxsxs) \\ &= xx((1_{\mathcal{U}} \otimes sss)xx) = xx((sss \otimes 1_{\mathcal{U}})xx) = sxsxs \otimes xx \\ &= s \otimes xx = (s \otimes 1_{\mathcal{U}})xx = (1_{\mathcal{U}} \otimes s)xx = 1_{\mathcal{U}} \otimes xsx = 1_{\mathcal{U}} \otimes x, \end{aligned}$$

and therefore, $x \in S$. As a consequence, if T is a $*$-subring of $\mathcal{U}$ contained in S and $t \in T$, note that $t^* \in T \subseteq S$ and $t^*t \in T \subseteq S$ is self-adjoint, so $(t^*t)^{[-1]} \in S$. Therefore, by Proposition 4.1.2(iv), $t^{[-1]} = (t^*t)^{[-1]}t^* \in S$.

In particular, since $\varphi(R)$ is contained in S, we see from the inductive construction of $\mathcal{R}(\varphi(R), \mathcal{U})$ that for each n, $\mathcal{R}_n(\varphi(R), \mathcal{U}) \subseteq S$, and therefore $\mathcal{U} \subseteq S$, i.e., $\mathcal{U} = S$. Thus, Proposition 4.1.11(2) tells us that φ is epic.

Assume conversely that φ is epic. Since φ factors through

$$R \to \mathcal{R}(\varphi(R), \mathcal{U}) \hookrightarrow \mathcal{U},$$

we deduce that the embedding $\mathcal{R}(\varphi(R), \mathcal{U}) \hookrightarrow \mathcal{U}$ is epic, and hence surjective by Proposition 4.1.14, i.e., $\mathcal{U} = \mathcal{R}(\varphi(R), \mathcal{U})$.

$\square$

By Corollary 3.1.15, an epic division R-ring $\mathcal{D}$ is uniquely determined by the Sylvester rank function it induces on R, i.e., by the values the rank $\mathrm{rk}_{\mathcal{D}}$ gives to matrices over (the image of) R. The next proposition, which is a particular case of [Jai19, Corollary 6.2] shows that a similar result holds in our situation.

Proposition 4.1.17. *Let R be a $*$-subring of a $*$-regular ring $\mathcal{U}$ such that $\mathcal{U} = \mathcal{R}(R, \mathcal{U})$ (equivalently, the embedding $R \hookrightarrow \mathcal{U}$ is epic). For every matrix A over $\mathcal{U}$, there exist matrices B_1, B_2 over R such that for every Sylvester matrix rank function* $\mathrm{rk} \in \mathbb{P}(\mathcal{U})$,

$$\mathrm{rk}(A) = \mathrm{rk}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} - \mathrm{rk}(B_1)$$

We can now introduce epic $*$-regular rings and isomorphisms between them, and use Corollary 4.1.15 and Proposition 4.1.17 to show how can they be characterized (Compare with the definition of an epic division ring and Cohn's result in terms of integer-valued rank functions).

Definition 4.1.18. Let R be a $*$-ring. An *epic $*$-regular R-ring* is a triple $(\mathcal{U}, \mathrm{rk}, \varphi)$ such that

(1.) $\mathcal{U}$ is a $*$-regular ring.

(2.) rk is a faithful Sylvester matrix rank function on $\mathcal{U}$.

(3.) $\varphi : R \to \mathcal{U}$ is a $*$-homomorphism.

(4.) $\mathcal{U} = \mathcal{R}(\varphi(R), \mathcal{U})$ (equivalently, by Proposition 4.1.16, φ is epic).

Two epic $*$-regular R-rings $(\mathcal{U}_1, \mathrm{rk}_1, \varphi_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, \varphi_2)$ are said to be *isomorphic* if there exists a $*$-isomorphism $\phi : \mathcal{U}_1 \to \mathcal{U}_2$ respecting the R-structure and the rank, i.e., such that the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{U}_1 \arrow[dd, "\phi"] \arrow[dr, "\mathrm{rk}_1"] & \\
R \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & \mathbb{R}_{\geq 0} \\
 & \mathcal{U}_2 \arrow[ur, "\mathrm{rk}_2"'] &
\end{tikzcd}$$

Remark. Recall that, since $\mathcal{U}_1$ and $\mathcal{U}_2$ are regular, rk_1 and rk_2 are determined by its values on elements (see Proposition 1.3.9) and therefore the equality $\mathrm{rk}_2(\phi(x)) = \mathrm{rk}_1(x)$ for every element $x \in \mathcal{U}_1$ is equivalent to $\mathrm{rk}_1 = \phi^\sharp(\mathrm{rk}_2)$. $\square$

Thus, in the same way that epic division R-rings are division rings—which are always equipped with their unique rank function—together with an epic homomorphism, epic $*$-regular R-rings (for a $*$-ring R) are $*$-regular rings on which we have a prescribed faithful rank function together with an epic $*$-homomorphism. The next result, due to A. Jaikin-Zapirain ([Jai19, Theorem 6.3]), is the analog of Cohn's result, and tells us that the values of the faithful rank function rk on matrices over the image of R uniquely determines the epic $*$-regular R-ring.

Theorem 4.1.19. *Let R be a $*$-ring. Two epic $*$-regular R-rings $(\mathcal{U}, \mathrm{rk}, \varphi)$, $(\mathcal{U}', \mathrm{rk}', \varphi')$ are isomorphic if and only if, for every matrix A over R, we have*

$$\mathrm{rk}(\varphi(A)) = \mathrm{rk}'(\varphi'(A)).$$

Proof. If the epic $*$-regular R-rings are isomorphic then there exists a $*$-isomorphism of R-rings $\phi : \mathcal{U} \to \mathcal{U}'$ such that $\mathrm{rk} = \phi^\sharp(\mathrm{rk}')$, and hence for every matrix A over R,

$$\mathrm{rk}(\varphi(A)) = \mathrm{rk}'(\phi(\varphi(A))) = \mathrm{rk}'(\varphi'(A)).$$

Conversely, note that $\mathcal{U} \times \mathcal{U}'$ is a $*$-regular ring with the involution $(u, u')^* = (u^*, (u')^*)$ and that $\varphi_0 = (\varphi, \varphi') : R \to \mathcal{U} \times \mathcal{U}'$ is a $*$-homomorphism. Hence, we can consider $\mathcal{U}_0 = \mathcal{R}(\varphi_0(R), \mathcal{U} \times \mathcal{U}')$. Since φ and $\varphi_0 : R \to \mathcal{U}_0$ are epic by Proposition 4.1.16, we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\varphi_0"] \arrow[d, "\varphi"'] & \mathcal{U}_0 \arrow[d, "\pi_{\mathcal{U}}"] \\ \mathcal{U} \arrow[r, "\mathrm{id}_{\mathcal{U}}"'] & \mathcal{U} \end{tikzcd}$$

where $\pi_{\mathcal{U}}$ denotes the projection on $\mathcal{U}$, and therefore Corollary 4.1.15 tells us that $\pi_{\mathcal{U}}(\mathcal{U}_0) = \mathrm{id}_{\mathcal{U}}(\mathcal{U}) = \mathcal{U}$. Similarly, $\pi_{\mathcal{U}'}(\mathcal{U}_0) = \mathcal{U}'$.

Let us show that $\pi_{\mathcal{U}}$ and $\pi_{\mathcal{U}'}$ are also injective. By our hypothesis, the rank functions $(\pi_{\mathcal{U}})^\sharp(\mathrm{rk})$ and $(\pi_{\mathcal{U}'})^\sharp(\mathrm{rk}')$ on $\mathcal{U}_0$ satisfy, for every matrix A over R,

$$\begin{aligned} [(\pi_{\mathcal{U}})^\sharp(\mathrm{rk})](\varphi_0(A)) &= \mathrm{rk}(\pi_{\mathcal{U}}\varphi_0(A)) = \mathrm{rk}(\varphi(A)) = \mathrm{rk}'(\varphi'(A)) \\ &= \mathrm{rk}'(\pi_{\mathcal{U}'}\varphi_0(A)) = [(\pi_{\mathcal{U}'})^\sharp(\mathrm{rk}')](\varphi_0(A)), \end{aligned}$$

and hence from Proposition 4.1.17, $(\pi_{\mathcal{U}})^\sharp(\mathrm{rk}) = (\pi_{\mathcal{U}'})^\sharp(\mathrm{rk}')$ as rank functions on $\mathcal{U}_0$. Therefore, if we take any $(u, u') \in \mathcal{U}_0$,

$$\mathrm{rk}(u) = \mathrm{rk}(\pi_{\mathcal{U}}(u, u')) = \mathrm{rk}'(\pi_{\mathcal{U}'}(u, u')) = \mathrm{rk}'(u').$$

Since rk' is faithful, if $\pi_{\mathcal{U}}(u, u') = u = 0$, then $0 = \mathrm{rk}(u) = \mathrm{rk}'(u')$ and hence $u' = 0$. Therefore $\pi_{\mathcal{U}}$ is injective (and similarly for $\pi_{\mathcal{U}'}$ using faithfulness of rk) and thus a ring isomorphism.

In fact, by definition of the involution in $\mathcal{U}_0$, they are $*$-isomorphisms, and consequently the composition $\pi_{\mathcal{U}'} \circ \pi_{\mathcal{U}}^{-1} : \mathcal{U} \to \mathcal{U}'$ is a $*$-isomorphism with

$$\pi_{\mathcal{U}'} \circ \pi_{\mathcal{U}}^{-1} \circ \varphi = \pi_{\mathcal{U}'} \circ \varphi_0 = \varphi'$$

and $\mathrm{rk} = \mathrm{rk}' \circ \pi_{\mathcal{U}'} \circ \pi_{\mathcal{U}}^{-1}$ because $(\pi_{\mathcal{U}})^\sharp(\mathrm{rk}) = (\pi_{\mathcal{U}'})^\sharp(\mathrm{rk}')$. The epic $*$-regular rings $(\mathcal{U}, \mathrm{rk}, \varphi)$ and $(\mathcal{U}', \mathrm{rk}', \varphi')$ are then isomorphic. □

The previous theorem motivates the following definition and gives us the subsequent corollary.

Definition 4.1.20. Let R be a $*$-ring and let rk be a Sylvester matrix rank function on R. We say that rk is $*$*-regular* if there exist a $*$-regular ring $\mathcal{U}$, a $*$-homomorphism $\varphi : R \to \mathcal{U}$ and a Sylvester matrix rank function rk' on $\mathcal{U}$ such that $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}')$. We denote by $\mathbb{P}_{*\mathrm{reg}}(R)$ the set of all $*$-regular Sylvester matrix rank functions on R.

Corollary 4.1.21. *Let R be a $*$-ring and let* rk *be a $*$-regular Sylvester matrix rank function on R. Then, there exists an epic $*$-regular R-ring $(\mathcal{U}, \mathrm{rk}', \varphi)$, unique up to isomorphism of epic $*$-regular R-rings, such that* $\mathrm{rk} = \varphi^\sharp(\mathrm{rk}')$.

Proof. Consider a $*$-regular ring $\mathcal{U}_0$, a $*$-homomorphism $\varphi_0 : R \to \mathcal{U}_0$ and a Sylvester matrix rank function rk_0 on $\mathcal{U}_0$ such that $\mathrm{rk} = (\varphi_0)^\sharp(\mathrm{rk}_0)$. As we have seen in Lemma 1.3.11, $\ker \mathrm{rk}_0$ is a two-sided ideal of $\mathcal{U}_0$, and if $\pi : \mathcal{U}_0 \to \mathcal{U}_0/\ker \mathrm{rk}_0$ is the quotient map, then rk_0 induces a faithful Sylvester matrix rank function rk_1 on $\mathcal{U}_1 := \mathcal{U}_0/\ker \mathrm{rk}_0$ such that $\mathrm{rk}_0 = \pi^\sharp(\mathrm{rk}_1)$. Moreover, by Corollary 4.1.4, $\mathcal{U}_1$ is also $*$-regular with involution $(x + I)^* = x^* + I$, and hence π is a $*$-homomorphism. Thus, the composition $\varphi_1 = \pi \circ \varphi_0 : R \to \mathcal{U}_1$ is a $*$-homomorphism, and $\mathrm{rk} = \varphi_1^\sharp(\mathrm{rk}_1)$ with rk_1 faithful. Finally, setting $\mathcal{U} = \mathcal{R}(\varphi_1(R), \mathcal{U}_1)$ and considering the restriction of rk_1 to $\mathcal{U}$ (see also Lemma 4.1.10), we have that $(\mathcal{U}, \mathrm{rk}_1, \varphi_1)$ is an epic $*$-regular R-ring with $\mathrm{rk} = \varphi_1^\sharp(\mathrm{rk}_1)$.

If $(\mathcal{U}', \mathrm{rk}_2, \varphi_2)$ is another epic $*$-regular R-ring with $\mathrm{rk} = \varphi_2^\sharp(\mathrm{rk}_2)$, then $\varphi_1^\sharp(\mathrm{rk}_1) = \varphi_2^\sharp(\mathrm{rk}_2)$ and the result follows from Theorem 4.1.19. □

Definition 4.1.22. Let R be a $*$-ring and rk a $*$-regular rank function on R. We say that the unique epic $*$-regular R-ring $(\mathcal{U}, \mathrm{rk}', \varphi)$ in Corollary 4.1.21 is the $*$*-regular envelope* of rk.

The same proof of the fact that $\mathbb{P}_{\mathrm{reg}}(R)$ is a compact convex subset of $\mathbb{P}(R)$ applies to $\mathbb{P}_{*\mathrm{reg}}$. We include it to address the main differences.

Proposition 4.1.23. *$\mathbb{P}_{*reg}(R)$ is a compact convex subset of $\mathbb{P}(R)$. In particular it is closed.*

Proof. Consider the set of elements of $\mathbb{P}_{*\mathrm{reg}}(R)$ indexed by some index set I. For every $i \in I$, let $(\mathcal{U}_i, \mathrm{rk}_i', \varphi_i)$ be the epic $*$-regular envelope of rk_i. The ring $\mathcal{U} = \prod_{i\in I} \mathcal{U}_i$ is $*$-regular with the component-wise involution, what makes the canonical projections $\pi_i : \mathcal{U} \to \mathcal{U}_i$ and the ring homomorphism $\varphi : R \to \mathcal{U}$ given by $\varphi(r) = (\varphi_i(r))_{i\in I}$ be $*$-homomorphisms. Since for every $i \in I$, we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\varphi_i"'] & \mathcal{U} \arrow[d, "\pi_i"] \\ & \mathcal{U}_i \end{tikzcd}$$

we see that every $*$-regular rank function on R comes from $\mathcal{U}$. Hence the convex-linear continuous map $\varphi^\sharp : \mathbb{P}(\mathcal{U}) \to \mathbb{P}(R)$ satisfies that $\mathrm{im}\, \varphi^\sharp = \mathbb{P}_{*\mathrm{reg}}(R)$. Since $\varphi^\sharp$ is convex-linear, its image is a convex set, and since $\varphi^\sharp$ is continuous and $\mathbb{P}(\mathcal{U})$ is compact, then its image is also compact. As a compact set of a Hausdorff space, it is closed. □

The notion of $*$-regular rank and its corresponding $*$-regular envelope, together with the notion of Hughes-free rank function introduced in Chapter 3 will play an important role in the proof of the strong Atiyah conjecture. As a consequence, we are interested in studying conditions under which the natural extension of a $*$-regular rank remains $*$-regular. For this purpose, we need to impose an additional property to the involution, namely, it needs not only be proper but positive definite.

Definition 4.1.24. We say that a $*$-ring R is *positive definite* (or that the involution is positive definite) if for every $n \in \mathbb{N}$ and for every $x_1, \dots, x_n \in R$, the equality $\sum_{i=1}^n x_i^* x_i = 0$ implies $x_i = 0$ for $i = 1, \dots, n$.

Note that, in general, if R is a $*$-ring, then we can define an involution on $\mathrm{Mat}_n(R)$ by setting, for a matrix $A = (a_{ij}) \in \mathrm{Mat}_n(R)$, $A^* = (a_{ji}^*)$. The previous condition on the involution of R is equivalent to the condition that the involution of $\mathrm{Mat}_n(R)$ is proper for every n.

Lemma 4.1.25. *Let R be a $*$-ring. Then R is positive definite if and only if the induced involution on $\mathrm{Mat}_n(R)$ is proper for every n. In particular, $\mathcal{U}$ is a positive definite $*$-regular ring if and only if $\mathrm{Mat}_n(R)$ is $*$-regular for every n.*

Proof. Assume first that R is positive definite and take a matrix $A = (a_{ij})$ in $\mathrm{Mat}_n(R)$. The ii-entry in A^*A is $\sum_{j=1}^n a_{ji}^* a_{ji}$, and hence if $A^*A = 0$, then we deduce that $a_{ij} = 0$ for every i and j, i.e., $A = 0$. Conversely, assume that the involution in $\mathrm{Mat}_n(R)$ is proper for every n. If $a_1, \dots, a_n \in R$ are such that $\sum_{i=1}^n a_i^* a_i = 0$, then the matrix A whose first column is given by $(a_1, \dots, a_n)^T$ and is zero everywhere else satisfies $A^*A = 0$, and hence $A = 0$, i.e., $a_i = 0$ for every i. The last statement follows from the fact that matrix rings over regular rings are regular (see Example 1.3.2). □

The next proposition gives us a first situation in which the natural extension is $*$-regular. Observe that here we do not only prove $*$-regularity (what could also be done using Proposition 4.1.23) but provide a $*$-regular ring $\mathcal{P}_{\omega,\tau}^{\mathcal{U}}$ from where the natural extension comes and that will be frequently used in the following. To start with, observe that if R is a $*$-ring and τ is a $*$-automorphism, then we can define an involution in $R[t^{\pm 1}; \tau]$ by setting $t^* = t^{-1}$, hence $(at)^* = t^{-1}a^*$, and extending this by linearity. Indeed, since τ is a $*$-automorphism this is well-defined (i.e., compatible with the twist), because

$$(ta)^* = a^* t^{-1} = t^{-1}\tau(a^*) = t^{-1}\tau(a)^* = (\tau(a)t)^*,$$

it is linear by definition, has order 2 (i.e. $(p^*)^* = p$) and satisfies

$$\begin{aligned} [(at^i)(bt^j)]^* &= [a\tau^i(b)t^{i+j}]^* = t^{-(i+j)}[a\tau^i(b)]^* = t^{-(i+j)}\tau^i(b)^*a^* \\ &= t^{-(i+j)}\tau^i(b^*)a^* = (t^{-j}b^*)(t^{-i}a^*) = (bt^j)^*(at^i)^*. \end{aligned}$$

With this involution, we have the following.

Proposition 4.1.26. *Let $\mathcal{U}$ be a positive definite $*$-regular ring, τ a $*$-automorphism of $\mathcal{U}$ and* rk *a τ-compatible Sylvester matrix rank function on $\mathcal{U}$. Consider the previous* *involution in $\mathcal{U}[t^{\pm 1};\tau]$. Then the natural transcendental extension $\widetilde{\mathrm{rk}}$ of rk to $\mathcal{U}[t^{\pm 1};\tau]$ is $*$-regular.*

Proof. Since the involution in $\mathcal{U}$ is positive definite, the previous lemma tells us that $S_n = \mathrm{Mat}_n(\mathcal{U})$, with the $*$-transpose involution, is $*$-regular for every $n \geq 1$. Therefore, the ring $S = \prod_{n=1}^{\infty} S_n$ is also $*$-regular with the component-wise involution. Fix a non-principal ultrafilter ω on $\mathbb{N}$ and let $\pi_n : S \to S_n$ be the canonical homomorphism onto the n-th factor. In S_n we have the rank $\frac{1}{n}\mathrm{rk}$, and hence by Proposition 1.4.14 we can define a rank function $\mathrm{rk}_\omega = \lim_\omega \frac{1}{n}\mathrm{rk}$ on S given by

$$\mathrm{rk}_\omega(B) = \lim_\omega \frac{\mathrm{rk}(\pi_n(B))}{n}$$

for every matrix B over S.

Consider the ring homomorphism $\psi_n : \mathcal{U}[t;\tau] \to \mathrm{Mat}_n(\mathcal{U})$ given in Eq. (1.1), that sends the polynomial p to the matrix associated to ϕ_n^p with respect to the canonical basis in $\mathrm{End}_{\mathcal{U}}(\mathcal{U}[t;\tau]/\mathcal{U}[t;\tau]t^n)$. Since rk is τ-compatible, it makes sense to talk about the natural extension $\widetilde{\mathrm{rk}}$ of rk to $\mathcal{U}[t;\tau]$, which is given by

$$\widetilde{\mathrm{rk}}(A) = \lim_{n\to\infty} \widetilde{\mathrm{rk}}_n(A) = \lim_{n\to\infty} \frac{\mathrm{rk}(\psi_n(A))}{n}$$

for every matrix A over $\mathcal{U}[t;\tau]$. Therefore, if we define the ring homomorphism $\psi = (\psi_n) : \mathcal{U}[t;\tau] \to S$ with $\psi(p) = (\psi_n(p))$, we can see that $\widetilde{\mathrm{rk}} = \psi^\sharp(\mathrm{rk}_\omega)$. Indeed, for a matrix A over $\mathcal{U}[t;\tau]$,

$$\begin{aligned}\psi^\sharp(\mathrm{rk}_\omega)(A) &= \mathrm{rk}_\omega(\psi(A)) = \lim_\omega \frac{\mathrm{rk}(\pi_n(\psi(A)))}{n} = \lim_\omega \frac{\mathrm{rk}(\psi_n(A))}{n} \\ &= \lim_{n\to\infty} \frac{\mathrm{rk}(\psi_n(A))}{n} = \widetilde{\mathrm{rk}}(A).\end{aligned}$$

Although we shall not need it, note in particular that the result is independent of the chosen ultralimit (see Proposition 1.4.13). Using Lemma 1.3.11 and Corollary 4.1.4, we obtain that $\ker \mathrm{rk}_\omega$ is a two-sided ideal of S, $\mathcal{P}^{\mathcal{U}}_{\omega,\tau} := S/\ker \mathrm{rk}_\omega$ is a $*$-regular ring and rk_ω induces a faithful rank function rk'_ω on $\mathcal{P}^{\mathcal{U}}_{\omega,\tau}$. Moreover, if $\pi' : S \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}$ is the natural map to the quotient and we put $f_\omega = \pi' \circ \psi : \mathcal{U}[t;\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}$, then we have that

$$f^\sharp_\omega(\mathrm{rk}'_\omega) = \psi^\sharp \pi'^\sharp(\mathrm{rk}'_\omega) = \psi^\sharp(\mathrm{rk}_\omega) = \widetilde{\mathrm{rk}}$$

In particular, $\mathrm{rk}'_\omega(f_\omega(t)) = \widetilde{\mathrm{rk}}(t) = 1$, what implies that $f_\omega(t)$ is invertible in $\mathcal{P}^{\mathcal{U}}_{\omega,\tau}$ by Lemma 1.3.12. Consequently we can extend f_ω to a ring homomorphism

$$f_\omega : \mathcal{U}[t^{\pm 1};\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}.$$

We finish the proof by showing that f_ω is a $*$-homomorphism. To do so, observe from the $*$-compatibility of τ and the expression in Eq. (1.2), that for every $x \in \mathcal{U}$, $\psi_n(x^*) = \psi_n(x)^*$. Therefore,

$$(f_\omega(x))^* = (\psi_n(x))^*_n + \ker \mathrm{rk}_\omega = (\psi_n(x^*))_n + \ker \mathrm{rk}_\omega = f_\omega(x^*).$$

In addition, $\psi_n(t) = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}$, and hence

$$\psi_n(t)\psi_n(t)^* = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I_{n-1} & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $\frac{1}{n}\operatorname{rk}(I_n - \psi_n(t)\psi_n(t)^*) = \frac{1}{n}$, and as a consequence we have $(I_n)_n - \psi(t)\psi(t)^* \in \ker \operatorname{rk}_\omega$, i.e, $f_\omega(t)f_\omega(t)^* = 1_{\mathcal{P}^{\mathcal{U}}_{\omega,\tau}}$. Similarly, $f_\omega(t)^* f_\omega(t) = 1_{\mathcal{P}^{\mathcal{U}}_{\omega,\tau}}$, and finally

$$f_\omega(t)^* = f_\omega(t)^{-1} = f_\omega(t^{-1}) = f_\omega(t^*).$$

This implies that f_ω is a $*$-homomorphism, as we wanted to see. $\square$

Remark 4.1.27. If in the previous proposition the original Sylvester matrix rank function rk on $\mathcal{U}$ is faithful, the map $f_\omega : \mathcal{U}[t^{\pm 1};\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}$ is injective.

Let $p \in \mathcal{U}[t;\tau]$ be non-zero with first non-zero coefficient a_i. Then the matrix B_n associated to ϕ_n^p for every $n > i$ (see Eq. (1.2)), is non-zero (hence $\psi_n(p) \neq 0$) and satisfies $\frac{1}{n}\operatorname{rk}(B_n) \geq \frac{(n-i)\operatorname{rk}(a_i)}{n}$. Therefore,

$$\begin{aligned} \operatorname{rk}'_\omega(f_\omega(p)) &= \operatorname{rk}'_\omega((B_n)_n + \ker \operatorname{rk}'_\omega) = \lim_\omega \frac{\operatorname{rk}(B_n)}{n} \\ &\geq \lim_\omega \frac{(n-i)\operatorname{rk}(a_i)}{n} = \operatorname{rk}(a_i) > 0, \end{aligned}$$

where the last equality follows from the existence of the actual limit in n (see Proposition 1.4.13), and the last inequality follows because rk is faithful. Thus, $f_\omega(p)$ cannot be zero, and $f_\omega : \mathcal{U}[t;\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}$ is injective. Consequently, its extension $f_\omega : \mathcal{U}[t^{\pm 1};\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}$ is also injective. $\square$

Let us fix a non-principal ultrafilter ω on $\mathbb{N}$ and construct, for a $*$-regular ring $\mathcal{U}$, a $*$-automorphism τ of $\mathcal{U}$ and a τ-compatible Sylvester matrix rank function rk on $\mathcal{U}$, the $*$-homomorphism

$$f_\omega : \mathcal{U}[t^{\pm 1};\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tau}. \tag{4.1}$$

To simplify notation, denote also by rk_ω (instead of $\operatorname{rk}'_\omega$), the faithful rank function on $\mathcal{P}^{\mathcal{U}}_{\omega,\tau}$. We have just proved that the natural extension of rk to $\mathcal{U}[t^{\pm 1};\tau]$ is precisely $\widetilde{\operatorname{rk}} = f_\omega^\sharp(\operatorname{rk}_\omega)$, and hence the (unique) $*$-regular envelope of $\widetilde{\operatorname{rk}}$ is (isomorphic to)

$$(\mathcal{R}(f_\omega(\mathcal{U}[t^{\pm 1};\tau]), \mathcal{P}^{\mathcal{U}}_{\omega,\tau}), \operatorname{rk}_\omega, f_\omega).$$

We can state and prove the analog of Proposition 3.1.20 for $*$-regular rings (compare with [Jai19, Proposition 7.5] for Laurent polynomial rings).

Proposition 4.1.28. *Let R be a $*$-ring, τ a $*$-automorphism of R and* rk *a τ-compatible $*$-regular Sylvester matrix rank function on R. Let $(\mathcal{U}, \operatorname{rk}', \varphi)$ be the $*$-regular envelope of* rk.

1.) *There exists a $*$-automorphism $\tilde{\tau}$ of $\mathcal{U}$ such that $\tilde{\tau}\circ\varphi = \varphi\circ\tau$ and rk' is $\tilde{\tau}$-compatible. In particular, this induces an epic ring homomorphism $\tilde{\varphi} : R[t^{\pm 1};\tau] \to \mathcal{U}[t^{\pm 1};\tilde{\tau}]$ that extends φ.*

2.) *If $\widetilde{\mathrm{rk}}'$ is the natural extension of rk to $\mathcal{U}[t^{\pm 1};\tilde{\tau}]$, then $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}')$ is the natural extension of rk to $R[t^{\pm 1};\tau]$.*

3.) *If $\mathcal{U}$ is positive definite $*$-regular and we endow $R[t^{\pm 1};\tau]$ and $\mathcal{U}[t^{\pm 1};\tilde{\tau}]$ with the previous involution, then $\widetilde{\mathrm{rk}}$ is a $*$-regular Sylvester matrix rank function on $R[t^{\pm 1};\tau]$ with $*$-regular envelope*

$$(\mathcal{R}(f_\omega\circ\tilde{\varphi}(R[t^{\pm 1};\tau]), \mathcal{P}^{\mathcal{U}}_{\omega,\tilde{\tau}}), \mathrm{rk}'_\omega, f_\omega\circ\tilde{\varphi}).$$

Proof. Since τ is a $*$-automorphism of R and rk is τ-compatible, we have that $(\mathcal{U}, \mathrm{rk}', \varphi\circ\tau)$ is also a $*$-regular envelope of rk. Hence, by Corollary 4.1.21, there exists a $*$-automorphism $\tilde{\tau}$ of $\mathcal{U}$ such that the following diagram commutes

$$\begin{tikzcd} & \mathcal{U} \arrow[dd, dotted, "\exists\tilde{\tau}"'] \arrow[dr, "\mathrm{rk}'"] & \\ R \arrow[ur, "\varphi"] \arrow[dr, "\varphi\circ\tau"'] & & \mathbb{R}_{\geq 0} \\ & \mathcal{U} \arrow[ur, "\mathrm{rk}'"'] & \end{tikzcd}$$

In particular, we obtain from its commutativity that rk' is $\tilde{\tau}$-compatible and hence we can consider the natural extension $\widetilde{\mathrm{rk}}'$ of rk' to $\mathcal{U}[t^{\pm 1};\tilde{\tau}]$. Now, the same reasoning in the proof of Proposition 3.1.20 tells us that the map $\tilde{\varphi} : R[t^{\pm 1};\tau] \to \mathcal{U}[t^{\pm 1};\tilde{\tau}]$ sending $t \mapsto \varphi(r)$ and $t \mapsto t$ is an epic ring homomorphism that extends φ and that $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}')$.

To prove the third statement, observe that since φ is a $*$-homomorphism, the induced map $\tilde{\varphi}$ is also a $*$-homomorphism. Therefore, we have a $*$-homomorphism

$$f_\omega\circ\tilde{\varphi} : R[t^{\pm 1};\tau] \to \mathcal{P}^{\mathcal{U}}_{\omega,\tilde{\tau}}$$

such that $\widetilde{\mathrm{rk}} = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}') = \tilde{\varphi}^\sharp f_\omega^\sharp(\mathrm{rk}'_\omega) = (f_\omega\circ\tilde{\varphi})^\sharp(\mathrm{rk}'_\omega)$ by the previous proposition. Since rk'_ω is faithful, the result follows. $\square$

As a consequence of Proposition 4.1.28 and Proposition 1.5.6 we have the following corollary.

Corollary 4.1.29. *Let R be a $*$-ring, τ a $*$-automorphism of R and $\{\mathrm{rk}_{(i)}\}$ a family of τ-compatible $*$-regular rank functions on R. For every $i\in\mathbb{N}$, let $\widetilde{\mathrm{rk}}_{(i)}$ be the natural extension of $\mathrm{rk}_{(i)}$ to $R[t^{\pm 1};\tau]$. Then, for every non-principal ultrafilter ω on $\mathbb{N}$, $\lim_\omega \widetilde{\mathrm{rk}}_{(i)}$, as a rank function on $R[t^{\pm 1};\tau]$ is the natural extension of $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_{(i)}$ as a rank function on R.*

Proof. Let $(\mathcal{U}_i, \mathrm{rk}'_{(i)}, \varphi_i)$ be the $*$-regular envelope of $\mathrm{rk}_{(i)}$. Then $\mathcal{U} = \prod_{i=1}^{\infty} \mathcal{U}_i$ is $*$-regular, $\varphi = (\varphi_i) : R \to \mathcal{U}$ is a $*$-homomorphism and if $\pi_i : \mathcal{U} \to \mathcal{U}_i$ is the natural projection onto $\mathcal{U}_i$, we have $\pi_i \circ \varphi = \varphi_i$. In particular, the Sylvester matrix rank function rk'_ω on $\mathcal{U}$ (see Proposition 1.4.14) given by

$$\mathrm{rk}'_\omega(B) = \lim_\omega \mathrm{rk}'_{(i)}(\pi_i(B))$$

for every matrix B over $\mathcal{U}$, satisfies that, if A is a matrix over R, then

$$\begin{aligned}\varphi^\sharp(\mathrm{rk}'_\omega)(A) &= \mathrm{rk}'_\omega(\varphi(A)) = \lim_\omega \mathrm{rk}'_{(i)}(\pi_i(\varphi(A))) \\ &= \lim_\omega \mathrm{rk}'_{(i)}(\varphi_i(A)) = \lim_\omega \mathrm{rk}_{(i)}(A),\end{aligned}$$

i.e., $\varphi^\sharp(\mathrm{rk}'_\omega) = \mathrm{rk}_\omega$. Thus, we have shown that rk_ω is $*$-regular, and it is also τ-compatible because each $\mathrm{rk}_{(i)}$ has this property.

Also by Proposition 4.1.28(1) and (2), there exists a $*$-automorphism $\tilde{\tau}_i$ of $\mathcal{U}_i$ such that $\tilde{\tau}_i \circ \varphi_i = \varphi_i \circ \tau$, $\mathrm{rk}'_{(i)}$ is $\tilde{\tau}_i$-compatible and the homomorphism $\tilde{\varphi}_i : R[t^{\pm 1}; \tau] \to \mathcal{U}[t^{\pm 1}; \tilde{\tau}_i]$ induced by φ_i satisfies $\widetilde{\mathrm{rk}}_{(i)} = \tilde{\varphi}_i^\sharp(\widetilde{\mathrm{rk}}'_{(i)})$. Therefore, $\tilde{\tau} = (\tilde{\tau}_i) : \mathcal{U} \to \mathcal{U}$ given by $\tilde{\tau}((x_i)_i) = (\tilde{\tau}_i(x_i))_i$, defines a $*$-automorphism of $\mathcal{U}$ with $\tilde{\tau} \circ \varphi = \varphi \circ \tau$, $\tilde{\tau}_i \circ \pi_i = \pi_i \circ \tilde{\tau}$ and rk'_ω is $\tilde{\tau}$-compatible because

$$\begin{aligned}\mathrm{rk}'_\omega(\tilde{\tau}(B)) &= \lim_\omega \mathrm{rk}'_{(i)}(\pi_i(\tilde{\tau}(B))) = \lim_\omega \mathrm{rk}'_{(i)}(\tilde{\tau}_i(\pi_i(B))) \\ &= \lim_\omega \mathrm{rk}'_{(i)}(\pi_i(B)) = \mathrm{rk}'_\omega(B).\end{aligned}$$

Summing up, this implies that we can form a commutative diagram

$$\begin{tikzcd} R[t^{\pm 1}; \tau] \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\tilde{\varphi}_i"'] & \mathcal{U}[t^{\pm 1}; \tilde{\tau}] \arrow[d, "\tilde{\pi}_i"] \\ & \mathcal{U}_i[t^{\pm 1}; \tilde{\tau}_i] \end{tikzcd}$$

where $\tilde{\varphi}$ and $\tilde{\pi}_i$ are the homomorphisms induced by φ and π_i, respectively. Since rk'_ω is $\tilde{\tau}$-compatible, there exists its natural extension $\widetilde{\mathrm{rk}}'_\omega$ to $\mathcal{U}[t^{\pm 1}; \tilde{\tau}]$ and moreover, as in the proof of Proposition 3.1.20, since $\mathrm{rk}_\omega = \varphi^\sharp(\mathrm{rk}'_\omega)$ one can deduce that $\widetilde{\mathrm{rk}}_\omega = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}'_\omega)$.

Now, set $\mathrm{rk}_0 := \lim_\omega \tilde{\pi}_i^\sharp(\widetilde{\mathrm{rk}}'_{(i)})$. We claim that $\mathrm{rk}_0 = \widetilde{\mathrm{rk}}'_\omega$. Indeed, rk_0 is a Sylvester matrix rank function on $\mathcal{U}[t^{\pm 1}; \tau]$ such that, for every matrix B over $\mathcal{U}$,

$$\mathrm{rk}_0(B) = \lim_\omega \widetilde{\mathrm{rk}}'_{(i)}(\tilde{\pi}_i(B)) = \lim_\omega \mathrm{rk}'_{(i)}(\pi_i(B)) = \mathrm{rk}'_\omega(B),$$

where the second equality follows because $\tilde{\pi}_i(B)$ is actually a matrix over $\mathcal{U}_i$ and $\widetilde{\mathrm{rk}}'_{(i)}$ extends $\mathrm{rk}'_{(i)}$, and

$$\mathrm{rk}_0(I_n + Bt) = \lim_\omega \widetilde{\mathrm{rk}}'_{(i)}(\tilde{\pi}_i(I_n + Bt)) = \lim_\omega \widetilde{\mathrm{rk}}'_{(i)}(I_n + \pi_i(B)t)) = \lim_\omega n = n,$$

where we have applied Proposition 1.5.6 to compute the latter rank. Another application of the same proposition gives us the claim, and thus, as we wanted to see, for every matrix A over $R[t^{\pm 1};\tau]$,

$$\begin{aligned}\widetilde{\mathrm{rk}}_\omega(A) &= \widetilde{\mathrm{rk}}'_\omega(\tilde\varphi(A)) = \mathrm{rk}_0(\tilde\varphi(A)) = \lim_\omega \widetilde{\mathrm{rk}}'_{(i)}(\tilde\pi_i\tilde\varphi(A)) \\ &= \lim_\omega \widetilde{\mathrm{rk}}'_{(i)}(\tilde\varphi_i(A)) = \lim_\omega \widetilde{\mathrm{rk}}_{(i)}(A),\end{aligned}$$

i.e., $\widetilde{\mathrm{rk}}_\omega = \lim_\omega \widetilde{\mathrm{rk}}_{(i)}$. □

Remark. While working on [JL20] we needed this result for the proof of the Lück's approximation conjecture (see Proposition 5.2.10), and for this we followed the way in which its non-skew version [Jai19, Corollary 7.8] was proved, i.e., we extended the results in Section 1.5 from Laurent polynomial rings (as developed in [Jai19]) to skew Laurent polynomial rings. After the existence of the natural transcendental extension being settled in general in [JiLi21], this corollary became a particular case of the more general continuity results [JiLi21, Theorem 1.2 & Example 7.3], and holds without any assumption on the $*$-structure of R and τ or the $*$-regularity of the rank functions. □

Let us finish showing how Hughes-freeness of a rank function and positive definite $*$-regular rings are going to be used in the proof of the Atiyah conjecture.

4.1.1 Using the Hughes-free rank condition

Let G be a locally indicable group, H a non-trivial finitely generated subgroup of G and $N \triangleleft H$ a normal subgroup with H/N infinite cyclic. If K is a subfield of $\mathbb{C}$ closed under complex conjugation, then the group ring $K[G]$ is a $*$-ring with involution given by extending linearly $(ag)^* = \bar a g^{-1}$ for $a \in K$, $g \in G$. Assume that we have a rank function rk on $K[G]$ satisfying:

1. rk is $*$-regular and its $*$-regular envelope $(\mathcal{U}, \mathrm{rk}', \phi)$ is positive definite.

2. rk is a Hughes-free Sylvester matrix rank function on $K[G]$.

Note that $K[N]$ and $K[H]$ are $*$-rings with the induced involution and hence it makes sense to consider $\mathcal{U}_N = \mathcal{R}(\phi(K[N]), \mathcal{U})$ and $\mathcal{U}_H = \mathcal{R}(\phi(K[H]), \mathcal{U})$. It follows then that rk_N and rk_H, the respective restrictions of rk to $K[N]$ and $K[H]$, are $*$-regular rank functions with $*$-regular envelopes $(\mathcal{U}_N, \mathrm{rk}'_N, \phi)$ and $(\mathcal{U}_H, \mathrm{rk}'_H, \phi)$, where rk'_N and rk'_H also denote the corresponding restrictions of rk'.

Take $x \in H$ such that $H/N = \langle Nx\rangle$ and let τ be the automorphism of $K[N]$ given by conjugation by x, so that $\tau(z) = xzx^{-1}$ for every $z \in K[N]$. Then τ is in fact a $*$-automorphism, because for every $n \in N$ we have the equality $(xnx^{-1})^* = (xnx^{-1})^{-1} = xn^{-1}x^{-1}$ in $K[N]$, and hence

$$\tau(z)^* = (xzx^{-1})^* = xz^*x^{-1} = \tau(z^*).$$

Moreover, since x is an invertible element in $K[H]$ and rk_N is just the restriction of rk, we must have for every $n \times m$ matrix A over $K[N]$ that

$$\mathrm{rk}_N(\tau(A)) = \mathrm{rk}((xI_n)A(x^{-1}I_m)) = \mathrm{rk}(A) = \mathrm{rk}_N(A).$$

Therefore, we are in the conditions of Proposition 4.1.28 for rk_N. Fixing a non-principal ultrafilter ω on $\mathbb{N}$, and since $\mathcal{U}_N$ is also positive definite, it tells us that there exists a $*$-automorphism $\tilde{\tau}$ of $\mathcal{U}_N$, an epic $*$-homomorphism $\tilde{\phi} : K[N][t^{\pm 1};\tau] \to \mathcal{U}_N[t^{\pm 1};\tilde{\tau}]$, a $*$-regular ring $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}$ with a faithful rank function rk'_ω and an injective $*$-homomorphism (see Eq. (4.1) and Remark 4.1.27)

$$f_\omega : \mathcal{U}_N[t^{\pm 1};\tilde{\tau}] \to \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}$$

such that the $*$-regular envelope of $\widetilde{\mathrm{rk}}_N$ is

$$(\mathcal{R}(f_\omega \circ \tilde{\phi}(K[N][t^{\pm 1};\tau]), \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}), \mathrm{rk}'_\omega, f_\omega \circ \tilde{\phi}).$$

Now, we observe the following.

- On the one hand, we have a $*$-isomorphism $\iota : K[H] \to K[N][t^{\pm 1};\tau]$ that acts as the identity on $K[N]$ and sends $x \mapsto t$, and since rk is Hughes-free, $\mathrm{rk}_H = \iota^\sharp(\widetilde{\mathrm{rk}}_N)$. Therefore,

$$(\mathcal{R}(f_\omega \circ \tilde{\phi} \circ \iota(K[H]), \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}), \mathrm{rk}'_\omega, f_\omega \circ \tilde{\phi} \circ \iota).$$

  is another $*$-regular envelope of rk_H. Thus by Corollary 4.1.21, the previous epic $*$-regular ring is isomorphic to $(\mathcal{U}_H, \mathrm{rk}'_H, \phi)$ as an epic $*$-regular $K[H]$-ring. If we set $\mathcal{S} = \mathcal{R}(f_\omega \circ \tilde{\phi} \circ \iota(K[H]), \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}})$, this means that there exists a $*$-isomorphism $\varphi : \mathcal{U}_H \to \mathcal{S}$ such that the following commutes

$$\begin{tikzcd} & K[H] \arrow[dl, "\phi"'] \arrow[dr, "f_\omega \circ \tilde{\phi} \circ \iota"] & \\ \mathcal{U}_H \arrow[rr, "\varphi"'] & & \mathcal{S} \end{tikzcd} \tag{4.2}$$

  and $\varphi^\sharp(\mathrm{rk}'_\omega) = \mathrm{rk}'_H$. In particular, $\mathcal{U}_H$ can be identified with a $*$-regular subring of $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}$, and via this identification we can think that rk'_H is the restriction of rk'_ω. We have constructed so far an embedding

$$\mathcal{U}_H \hookrightarrow \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}.$$

- On the other hand, let $\mathcal{U}_N[[t;\tilde{\tau}]]$ denote the *skew power series ring*. The natural product (as for skew polynomials) is well-defined because for every degree there are only finitely many monomials to multiply. Note that the map $\psi : \mathcal{U}_N[t;\tilde{\tau}] \to \prod_{n=1}^{\infty} \mathrm{Mat}_n(\mathcal{U}_N)$ constructed in Proposition 4.1.28 actually extends to an injective ring homomorphism

$$\mathcal{U}_N[[t;\tilde{\tau}]] \hookrightarrow \prod_{n=1}^{\infty} \mathrm{Mat}_n(\mathcal{U}_N)$$

sending the formal power series p to the tuple of matrices associated to ϕ_n^p, the endomorphism in $\mathrm{End}_{\mathcal{U}}(\mathcal{U}[t;\tilde{\tau}]/\mathcal{U}[t;\tilde{\tau}]t^n)$ induced by right multiplication by p, with respect to the canonical basis. Similarly, this induces a ring homomorphism

$$f_\omega : \mathcal{U}_N[[t;\tilde{\tau}]] \to \mathcal{P}_{\omega,\tilde{\tau}}^{\mathcal{U}_N}.$$

and the same proof in Remark 4.1.27 shows that this f_ω is injective.

Observe that the powers of t in $\mathcal{U}_N[[t;\tilde{\tau}]]$ satisfy the left and right Ore condition because $\tilde{\tau}$ is an automorphism, and that the Ore localization of $\mathcal{U}_N[[t;\tilde{\tau}]]$ with respect to the powers of t is isomorphic to the *skew Laurent series ring* $\mathcal{U}_N((t;\tilde{\tau}))$, the ring whose elements are formal power series $\sum_{n\in\mathbb{Z}} a_n t^n$ in which only finitely many coefficients a_n for $n<0$ are non-zero. Thus, since $f_\omega(t)$ is invertible in $\mathcal{P}_{\omega,\tilde{\tau}}^{\mathcal{U}_N}$, f_ω extends to an embedding

$$f_\omega : \mathcal{U}_N((t;\tilde{\tau})) \hookrightarrow \mathcal{P}_{\omega,\tilde{\tau}}^{\mathcal{U}_N}$$

By definition of each f_ω, we have a commutative diagram of injective ring homomorphisms.

$$\begin{tikzcd}
\mathcal{U}_N[t;\tilde{\tau}] \arrow[rr] \arrow[dd] \arrow[dr, "f_\omega"] & & \mathcal{U}_N[[t;\tilde{\tau}]] \arrow[dd] \arrow[dl, "f_\omega"'] \\
& \mathcal{P}_{\omega,\tilde{\tau}}^{\mathcal{U}_N} & \\
\mathcal{U}_N[t^{\pm 1};\tilde{\tau}] \arrow[rr] \arrow[ur, "f_\omega"] & & \mathcal{U}_N((t;\tilde{\tau})) \arrow[ul, "f_\omega"']
\end{tikzcd} \tag{4.3}$$

□

Let us glue things together. Consider the diagram

$$\begin{tikzcd}
K[N] \arrow[rrr, shift left, "\phi"] \arrow[rrr, shift right] \arrow[ddd, shift right, "j"'] \arrow[ddd, shift left] & & & \mathcal{U}_N \arrow[dll, shift left] \arrow[dll, shift right, "j''"'] \arrow[ddd, shift right] \arrow[ddd, shift left, "j'"] \\
& \mathcal{U}_N((t;\tilde{\tau})) \arrow[r, shift left] \arrow[r, shift right, "f_\omega"'] & \mathcal{P}_{\omega,\tilde{\tau}}^{\mathcal{U}_N} & \\
& & & \\
K[H] \arrow[uur, shift left, "\tilde{\phi}\circ\iota"] \arrow[uur, shift right] \arrow[rrr, shift left] \arrow[rrr, shift right, "\phi"'] & & & \mathcal{U}_H \arrow[uul, shift left] \arrow[uul, shift right, "\varphi"]
\end{tikzcd}$$

where j, j', j'' are just the inclusion maps. The subdiagram in red commutes by definition, the subdiagram in orange commutes because $\tilde{\phi}\circ\iota$ is the homomorphim that acts as ϕ on elements of $K[N]$ and sends $x \mapsto t$, and the subdiagram in blue commutes as a consequence of the commutativity in Eq. (4.2) and Eq. (4.3). We claim that the black subdiagram also commutes. Indeed,

$$\varphi\circ j'\circ\phi = \varphi\circ\phi\circ j = f_\omega\circ\tilde{\phi}\circ\iota\circ j = f_\omega\circ j''\circ\phi,$$

where the color indicates the subdiagram we use. Thus, since the upper $\phi : K[N] \to \mathcal{U}_N$ is epic, we obtain that $\varphi \circ j' = f_\omega \circ j''$, as we wanted.

This diagram can then be broken into two commutative pieces. We first have the union of the orange and blue subdiagrams

$$\begin{tikzcd} K[N] \arrow[r, hook] \arrow[d, "\phi"'] & K[H] \arrow[r, "\phi"] \arrow[d, "\tilde{\phi}\circ\iota"'] & \mathcal{U}_H \arrow[d, "\varphi"] \\ \mathcal{U}_N \arrow[r, hook] & \mathcal{U}_N((t;\tilde{\tau})) \arrow[r, "f_\omega"'] & \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}} \end{tikzcd} \tag{4.4}$$

This kind of diagrams will be needed to apply induction once the complexity is introduced in Section 4.3.3. Secondly, we have just proved that the following commutes.

$$\begin{tikzcd} \mathcal{U}_N \arrow[r, hook] \arrow[d, hook] & \mathcal{U}_H \arrow[d, hook, "\varphi"] \\ \mathcal{U}_N((t;\tilde{\tau})) \arrow[r, hook, "f_\omega"'] & \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}} \end{tikzcd}$$

with $\varphi^\sharp(\mathrm{rk}'_\omega) = \mathrm{rk}'_H$. Thus, identifying $\mathcal{U}_H$ and $\mathcal{U}_N((t;\tilde{\tau}))$ with their images, we can consider the intersection $\mathcal{U}_H \cap \mathcal{U}_N((t;\tilde{\tau}))$ inside $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}$, and we shall use this to express elements of $\mathcal{U}_H$ as a formal power series whose coefficients, which are in $\mathcal{U}_N$, are "less complex" than the original element, allowing us to use induction.

4.2 $\mathcal{U}(G)$ and the strong Atiyah conjecture

The Atiyah conjecture was originally introduced as a question of rationality of certain L^2-Betti numbers associated to a group G, and it was later reformulated and generalized (what is usually attributed to W. Lück and T. Schick) in terms of the possible values that a prescribed Sylvester rank function on the group ring $K[G]$, for a subfield K of $\mathbb{C}$, may take (cf. [Lüc02, Chapter 10]). The goal of this section is to introduce the main objects taking part in the formulation of the (strong) Atiyah conjecture and its relation to the existence of a division $K[G]$-ring of fractions for a torsion-free group G. Nevertheless, although we sketch and/or comment some proofs of the results, this section is not intended to be a self-contained introduction to the topic, but rather to collect and unify some of the most basic properties that can be found in [Rei98], [Lüc02], [Jai19S], [Kam19], and several papers of P. Linnell and T. Schick on the topic (for instance, [Lin91, Lin93, Lin98, Lin06, Lin08, LS12, Sch00, Sch00*]).

Let G be a countable group, and let $\mathbb{C}[G]$ denote the group algebra with coefficients in G. The set

$$\ell^2(G) := \left\{ \sum_{g\in G} a_g g : a_g \in \mathbb{C} \text{ and } \sum_{g\in G} |a_g|^2 < \infty \right\}$$

of formal square summable series indexed by G with complex coefficients forms a Hilbert space with orthonormal basis G and inner product given by

$$\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g\rangle = \sum_{g\in G} a_g \bar{b}_g,$$

where $\bar{b}_g$ denotes the complex conjugate of b_g. Left and right multiplication by elements of G give rise to a left and a right action of G on $\ell^2(G)$ given by $h\cdot\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g hg$ and $\left(\sum_{g\in G} a_g g\right)\cdot h = \sum_{g\in G} a_g gh$, respectively, and these actions commute. We can linearly extend for instance the right action to $\mathbb{C}[G]$, so that we can see an element in $\mathbb{C}[G]$ acting as a bounded linear operator on the right of $\ell^2(G)$ that commutes with the left action of G. In other words, we can see $\mathbb{C}[G]$ as a subalgebra of the algebra $\mathcal{B}(\ell^2(G))$ of bounded linear operators on $\ell^2(G)$.

All of these objects so far come equipped with a $*$-operation. In $\mathcal{B}(\ell^2(G))$ the $*$-operation consists on taking adjoints, i.e., if $T : \ell^2(G) \to \ell^2(G)$ is a bounded linear operator, T^* denotes the unique bounded linear operator $T^* : \ell^2(G) \to \ell^2(G)$ with the property that, for all $x, y \in \ell^2(G)$,

$$\langle (y)T, z\rangle = \langle y, (z)T^*\rangle.$$

In $\ell^2(G)$ and $\mathbb{C}[G]$ the $*$-operation takes the element $x = \sum_{g\in G} a_g g$ to the element $x^* = \sum_{g\in G} \bar{a}_g g^{-1}$. This operation defines an involution (of $\mathbb{C}$-vector spaces) on $\ell^2(G)$ and makes $\mathbb{C}[G]$ a $*$-ring as introduced in the previous section. Moreover, if we have $x = \sum_{g\in G} a_g g \in \mathbb{C}[G]$ and $y = \sum_{g\in G} b_g g$, $z = \sum_{g\in G} c_g g \in \ell^2(G)$, then

$$\begin{aligned}\langle (y)x, z\rangle &= \langle \sum_{g\in G}\left(\sum_{hk=g} b_h a_k\right) g, z\rangle = \sum_{g\in G}\left(\sum_{hk=g} b_h a_k\right)\bar{c}_g \\ &= \sum_{h\in G} b_h \overline{\sum_{gk^{-1}=h} c_g \bar{a}_k} = \langle y, \sum_{h\in G}\left(\sum_{gk^{-1}=h} c_g \bar{a}_k\right) h\rangle = \langle y, (z)x^*\rangle.\end{aligned}$$

Therefore, x^* is precisely the adjoint of x when considered as an element of $\mathcal{B}(\ell^2(G))$, what makes $\mathbb{C}[G]$ a (unital) $*$-subalgebra of $\mathcal{B}(\ell^2(G))$. Whenever we have a unital $*$-closed subalgebra $\mathcal{A}$ of an algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, von Neumann bicommutant theorem tells us that the closure of $\mathcal{A}$ in the weak operator topology, the closure of $\mathcal{A}$ in the strong operator topology and the double commutant $\mathcal{A}''$ of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ coincide. In our case, the object arising from $\mathbb{C}[G]$ in any of the previous equivalent ways is the so-called *group von Neumann algebra* $\mathcal{N}(G)$, and hence $\mathcal{N}(G)$ is in particular a $*$-ring containing $\mathbb{C}[G]$ as a $*$-subring.

Another useful description of $\mathcal{N}(G)$ is that it is the algebra of bounded linear left G-equivariant (i.e., that commute with the left action of G on $\ell^2(G)$) operators on $\ell^2(G)$ (see, for instance, [Lin91, Lemma 5] or [Lin98, Section 8]), i.e.,

$$\mathcal{N}(G) = \left\{ f \in \mathcal{B}(\ell^2(G)) : (g\cdot x)f = g\cdot(x)f, \text{ for all } g\in G, x\in\ell^2(G)\right\}.$$

The group von Neumann algebra possesses a faithful normal trace

$$\mathrm{Tr} : \mathcal{N}(G) \to \mathbb{C}$$

given by $\mathrm{Tr}(f) = \langle (e)f, e\rangle$, where e denotes the neutral element of G, and hence it is an example of a finite von Neumann algebra. For instance, $\mathbb{C}$-*linearity* of Tr can be shown from the definition, and the *trace property* $\mathrm{Tr}(f_1f_2) = \mathrm{Tr}(f_2f_1)$ can be shown first for elements f_1, f_2 in $\mathbb{C}[G]$ and then extended to $\mathcal{N}(G)$ observing that Tr is weakly continuous. *Faithfulness* here means that for every element $f \in \mathcal{N}(G)$, $\mathrm{Tr}(ff^*) = 0$ if and only if $f = 0$. To show this, if $(e)f = \sum a_g g$, observe that

$$\langle (e)ff^*, e\rangle = \langle (e)f, (e)f\rangle = \sum_{g\in G} |a_g|^2$$

equals zero if and only if every a_g is zero, i.e., if and only if $(e)f = 0$. Since f is left G-equivariant, we obtain that $(x)f = x \cdot (e)f = 0$ for $x \in \mathbb{C}[G]$, and since $\mathbb{C}[G]$ is dense in $\mathcal{N}(G)$, this implies that $f = 0$. Observe from the previous expression that Tr is also *positive*, i.e., for any operator of the form $f' = ff^*$, we have that $\mathrm{Tr}(f')$ is a non-negative real number. Finally, *normality* means that Tr is continuous with respect to the ultraweak topology.

Note that while discussing faithfulness of Tr we have shown that an element $f \in \mathcal{N}(G)$ is zero if and only if $(e)f \in \ell^2(G)$ is zero. Therefore, the linear map $\mathcal{N}(G) \to \ell^2(G)$ sending $f \mapsto (e)f$ is injective and hence allows us to see $\mathcal{N}(G)$ as a subspace of $\ell^2(G)$. In addition, observe that if $x = \sum a_g g \in \ell^2(G)$, we can compute a_g as $\langle x, g\rangle$. If $f \in \mathcal{N}(G)$ is such that $(e)f = \sum a_g g \in \ell^2(G)$ and the adjoint operator f^* is such that $(e)f^* = \sum b_g g$, then

$$b_g = \langle (e)f^*, g\rangle = \langle e, (g)f\rangle = \langle e, g \cdot (e)f\rangle = \langle e, \sum_{h\in G} a_h gh\rangle = \bar{a}_{g^{-1}}.$$

Therefore, $(e)f^* = \sum \bar{a}_{g^{-1}} g = \sum \bar{a}_g g^{-1} = [(e)f]^*$. In other words, the previous embedding preserves the $*$-operation and we have a chain of $*$-embeddings

$$\mathbb{C}[G] \subseteq \mathcal{N}(G) \subseteq \ell^2(G)$$

We can now go a step further and introduce a ring $\mathcal{U}(G)$ which is usually quite larger than $\mathcal{N}(G)$ but also quite richer from the perspective of ring theory. This object $\mathcal{U}(G)$ is the *algebra of unbounded* (or maybe more precisely, not necessarily bounded) *operators on $\ell^2(G)$ affiliated to $\mathcal{N}(G)$*. This means that an element f of $\mathcal{U}(G)$ is a linear operator $f : \mathrm{dom}(f) \subseteq \ell^2(G) \to \ell^2(G)$ where $\mathrm{dom}(f)$ is dense in $\ell^2(G)$, f is closed (i.e., its graph is closed in $(\ell^2(G))^2$) and affiliated to $\mathcal{N}(G)$, what in our context means that f commutes with the left action of G on $\ell^2(G)$ (see [Rei98, Lemma 11.8 and the subsequent discussion]). In other words,

$$\mathcal{U}(G) = \left\{ f : \mathrm{dom}(f) \subseteq \ell^2(G) \to \ell^2(G) : \begin{array}{l} f \text{ linear and closed} \\ \mathrm{dom}(f) \text{ dense in } \ell^2(G) \\ g \cdot (x)f = (g \cdot x)f, \forall g \in G \end{array} \right\}$$

Observe that one has to be very careful when defining the operations and the involution in $\mathcal{U}(G)$. For example, the "naive" sum of two unbounded operators f_1 and f_2 is not necessarily closed and it is only defined in a set containing $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, which must be shown to be dense. A good source to learn the basic facts about $\mathcal{U}(G)$ is H. Reich's thesis [Rei98], and a survey on the basic properties of unbounded operators and $\mathcal{U}(G)$ can be found in [Lüc02, Subsection 1.4.1 and Section 8.1]. In the next proposition we just list the properties of $\mathcal{U}(G)$ that we shall use in the following, most of which are consequences of the availability of functional calculus (cf. [Rei98, Proposition 11.4]) and polar decomposition for closed densely defined operators (cf. [Rei98, Proposition 11.5]).

Here, the reader should be warned that the order in which the properties are listed do not obey the logical order in which they are proved but rather responds to the most condensed way of presenting them.

Proposition 4.2.1.

1. *$\mathcal{U}(G)$ is a positive definite $*$-regular ring containing $\mathcal{N}(G)$ as a $*$-subring. Moreover, every projection in $\mathcal{U}(G)$ lies in $\mathcal{N}(G)$.*

2. *The set of non-zero-divisors in $\mathcal{N}(G)$ satisfies the left and right Ore conditions and $\mathcal{U}(G)$ is isomorphic to the classical quotient ring of $\mathcal{N}(G)$.*

3. *If $f \in \mathcal{U}(G)$, then there exists a partial isometry $u \in \mathcal{N}(G)$ (i.e. uu^* is a projection) such that $\mathrm{RP}(f) = u^*u$ and $\mathrm{RP}(f^*) = \mathrm{LP}(f) = uu^*$.*

Proof. For the fact that $\mathcal{U}(G)$ is a $*$-ring, one can consult for instance [Rei98, Theorem 2.2 and Appendix I]. Briefly, for elements $f_1, f_2 \in \mathcal{U}(G)$, one defines $f_1 + f_2$ and $f_1 f_2$ as the closures of the natural (naive) operations, and f_1^* is defined as usual, i.e., as an operator $f_1^* : \mathrm{dom}(f_1^*) \subseteq \ell^2(G) \to \ell^2(G)$ satisfying

$$\langle (x)f_1, y\rangle = \langle x, (y)f_1^*\rangle$$

for every $x \in \mathrm{dom}(f_1)$. Here, $\mathrm{dom}(f_1^*)$ consists of those elements $y \in \ell^2(G)$ for which $\langle (\square)f_1, y\rangle$ is a continuous linear functional on $\mathrm{dom}(f_1)$.

A proof of $*$-regularity of $\mathcal{U}(G)$ can be found in [Rei98, Proposition 2.10 & Note 2.11], and positive definiteness follows for instance from [LS12, Lemma 2.5], where it is proved more generally that, for elements $f_1, \dots, f_n \in \mathcal{U}(G)$, one has $\mathcal{U}(G) f_i \subseteq \mathcal{U}(G)\left(\sum_{i=1}^n f_i^* f_i\right)$. The last statement in 1. follows from the fact that projections in $\mathcal{U}(G)$ are bounded.

In [Rei98, Proposition 2.8] it is shown that $\mathcal{N}(G)$ satisfies both Ore conditions with respect to the set of non-zero-divisors and that $\mathcal{U}(G)$ is isomorphic to its classical quotient ring.

For the last statement, note that $\mathcal{N}(G)$ fulfills the equivalent conditions in [Ber82, Theorem 5] by [Ber72, §4 Proposition 9 & §13 Corollary to Proposition 2] and [Ber82, Section 4], and hence this follows from the remark after [Ber82, Theorem 5].

∎

The existence of a ring with the previous properties can actually be proved for any finite AW* algebra $\mathcal{A}$, i.e., one can construct a positive definite $*$-regular ring $\mathbf{C}$ containing $\mathcal{A}$ as a $*$-subring for which property 3. holds and that coincides with the classical quotient ring of $\mathcal{A}$ (see [Ber72, Chapter 8] and [Ber82, Theorems 1 & 3, and the proof of Theorem 10]).

Following [Lin98, Section 8], we complete the chain of $*$-embeddings by noting that, for every $x \in \ell^2(G)$ the map $\mathbb{C}[G] \to \ell^2(G)$ given by right multiplication by x is a densely defined (because $\mathbb{C}[G]$ is dense in $\ell^2(G)$) operator which commutes with the left action of G on $\ell^2(G)$. Moreover, it can be shown to extend to a closed left G-equivariant operator (see the proof of [Lin98, Lemma 11.3]), and hence defines a unique element $\tilde{x}$ in $\mathcal{U}(G)$ (cf. [Rei98, Proposition 11.19]). Therefore, we have defined a map $\ell^2(G) \to \mathcal{U}(G)$, which is actually injective. This embedding extends the embedding $\mathcal{N}(G) \subseteq \mathcal{U}(G)$, because the map induced by right multiplication by $(e)f$ coincides with f in $\mathbb{C}[G]$, and hence f is its unique closed extension in $\mathcal{U}(G)$. Similarly, it is $*$-preserving because one can show that $(y)\tilde{x}^* = yx^*$ for every $y \in \mathbb{C}[G]$ and hence $\tilde{x}^*$ is the unique closed extension of x^* in $\mathcal{U}(G)$, i.e., $\tilde{x}^* = \widetilde{x^*}$. Summing up, we have the chain of $*$-embeddings

$$\mathbb{C}[G] \subseteq \mathcal{N}(G) \subseteq \ell^2(G) \subseteq \mathcal{U}(G).$$

The next step is to show that for a subgroup H of G we have a commutative diagram of $*$-ring embeddings

$$\begin{tikzcd} \mathbb{C}[H] \arrow[r] \arrow[d] & \mathcal{N}(H) \arrow[r] \arrow[d] & \mathcal{U}(H) \arrow[d] \\ \mathbb{C}[G] \arrow[r] & \mathcal{N}(G) \arrow[r] & \mathcal{U}(G). \end{tikzcd}$$

Note first that $\mathbb{C}[H]$ is naturally a $*$-subring of $\mathbb{C}[G]$ and $\ell^2(H)$ is a $*$-closed subspace of $\ell^2(G)$. Now, let T be a left transversal of H in G (i.e., T contains a unique representative $t \in G$ for each left coset gH, $g \in G$) containing the neutral element e. Since $\sum_{i=1}^n t_i x_i \in \ell^2(G)$ for all $t_i \in T$ and $x_i \in \ell^2(H)$ and T is a left transversal, we have that $S = \bigoplus_{t \in T} t\ell^2(H)$ is a $\mathbb{C}$-linear left G-invariant subspace of $\ell^2(G)$, which is also dense because it contains $\mathbb{C}[G]$. If we take an element $f \in \mathcal{N}(H)$, then we can define a linear left G-equivariant map $\hat{f} : S \to \ell^2(G)$ by setting $(\sum_{i=1}^n t_i x_i)\hat{f} = \sum_{i=1}^n t_i (x_i) f$. This can be shown to extend to an element $\bar{f} \in \mathcal{N}(G)$ such that the map $\mathcal{N}(H) \to \mathcal{N}(G)$ given by $f \mapsto \bar{f}$ is an injective ring homomorphism that does not depend on the choice of T (cf. [Lüc02, Section 1.1.5]).

The extension to $\mathcal{N}(G)$ of an element $x \in \mathbb{C}[H]$ is precisely the map given by right multiplication by x, and hence the left square commutes. Moreover, since $\bar{f}$ coincides with f when restricted to $\ell^2(H)$, one has $(e)f = (e)\bar{f}$, and hence the square

$$\begin{tikzcd} \mathcal{N}(H) \arrow[r] \arrow[d] & \ell^2(H) \arrow[d] \\ \mathcal{N}(G) \arrow[r] & \ell^2(G) \end{tikzcd}$$

also commutes. Because of this, we can show on the one hand that $\mathcal{N}(H) \to \mathcal{N}(G)$ is $*$-preserving, since it is injective and the other maps in the square are injective and $*$-preserving (cf. [Lüc02, Exercise 1.5]), and on the other hand that the trace of an element in $\mathcal{N}(H)$ does not depend on whether we see it in $\mathcal{N}(H)$ or in $\mathcal{N}(G)$ (cf. [Lüc02, Lemma 1.24(1)]), i.e.,

$$\mathrm{Tr}_{\mathcal{N}(H)}(f) = \mathrm{Tr}_{\mathcal{N}(G)}(\bar{f}).$$

The same idea can be used to define the embedding $\mathcal{U}(H) \to \mathcal{U}(G)$. More precisely, if $f \in \mathcal{U}(H)$ has domain $\mathrm{dom}(f)$, then the left G-equivariant map $\hat{f} : \bigoplus_{t\in T} t\,\mathrm{dom}(f) \to \ell^2(G)$ given by setting $(\sum_{i=1}^n t_i x_i)\hat{f} = \sum_{i=1}^n t_i (x_i) f$ can be extended to an element $\bar{f} \in \mathcal{U}(G)$, and again the induced map $\mathcal{U}(H) \to \mathcal{U}(G)$, $f \mapsto \bar{f}$ is an injective ring homomorphism that does not depend on the choice of T (cf. [Lüc02, Page 323, eq. (8.12)]). By construction, this makes the right square commutative and one can show that $\mathcal{U}(H) \to \mathcal{U}(G)$ is $*$-preserving by making use of the facts that $\mathcal{N}(H) \to \mathcal{N}(G) \to \mathcal{U}(G)$ is $*$-preserving and that every element in $\mathcal{U}(H)$ can be written in the form ab^{-1} for $a, b \in \mathcal{N}(H)$ by Proposition 4.2.1(2).

Let ι denote the embedding $\mathcal{U}(H) \to \mathcal{U}(G)$. If K is a subfield of $\mathbb{C}$ closed under complex conjugation (so that $K[H]$ is $*$-closed) and $\mathcal{R}_{K[H]}$ denotes the $*$-regular closure of $K[H]$ in $\mathcal{U}(H)$, then since ι is a $*$-isomorphism onto its image $\iota(\mathcal{U}(H))$, we deduce that

$$\iota(\mathcal{R}_{K[H]}) = \mathcal{R}(\iota(K[H]), \iota(\mathcal{U}(H))) = \mathcal{R}(\iota(K[H]), \mathcal{U}(G)) \subseteq \mathcal{R}_{K[G]},$$

where the last equality follows from Lemma 4.1.10. This is important, since we will later write $\mathcal{U}(H) \subseteq \mathcal{U}(G)$ without further comments, and this implies that the notation $\mathcal{R}_{K[H]}$ is not ambiguous, since it does not depend on whether we are considering $K[H]$ inside $\mathcal{U}(H)$ or $\mathcal{U}(G)$. We have shown that the following diagram also commutes

$$\begin{tikzcd}
K[H] \arrow[r] \arrow[d] & \mathcal{R}_{K[H]} \arrow[r] \arrow[d] & \mathcal{U}(H) \arrow[d] \\
K[G] \arrow[r] & \mathcal{R}_{K[G]} \arrow[r] & \mathcal{U}(G).
\end{tikzcd}$$

A similar argument invoking Lemma 3.3.3(2) instead of Lemma 4.1.10 gives the same diagram replacing $*$-regular closures by division closures of $K[H]$ and $K[G]$, for any subfield K of $\mathbb{C}$. As a final remark, which is implicit in the above argument, if $x \in \mathcal{U}(H)$ and $p = \mathrm{RP}(x)$, then $\mathcal{U}(H)x = \mathcal{U}(H)p$. Hence, $xp = x$ and there exists $u \in \mathcal{U}(H)$ with $ux = p$. Since the map ι is a $*$-homomorphism, $\iota(p)$ is a projection satisfying $\mathcal{U}(G)\iota(x) = \mathcal{U}(G)\iota(p)$, i.e., $\mathrm{RP}(\iota(x)) = \iota(p) = \iota(\mathrm{RP}(x))$. Analogously, $\mathrm{LP}(\iota(x)) = \iota(\mathrm{LP}(x))$, and hence it also preserves relative inverses, i.e., $\iota(x^{[-1]}) = \iota(x)^{[-1]}$.

After this construction, one important feature of $\mathcal{U}(G)$ is that it admits a faithful Sylvester matrix rank function rk_G. We give here a proof based on the existence of Tr and the properties of $*$-regular rings, following the lines of exposition in [Lin08, Property 3], and the reader may also consult [JiLi21, Proposition 2.4]. We later relate it to the dimension function $\dim_{\mathcal{U}(G)}$ defined in [Lüc02, Chapter 8] and [Rei98, Section 3.2].

Proposition 4.2.2. *Let $x \in \mathcal{U}(G)$ and let p be the unique projection such that $\mathcal{U}(G)x = \mathcal{U}(G)p$ (i.e., $p = \mathrm{RP}(x)$). Then $\mathrm{rk}_G(x) = \mathrm{Tr}(p)$ defines a faithful pseudo-rank function on $\mathcal{U}(G)$ satisfying $\mathrm{rk}_G(x) = \mathrm{rk}_G(x^*)$.*
Additionally, if H is a subgroup of G and we identify $\mathcal{U}(H) \subseteq \mathcal{U}(G)$, then $\mathrm{rk}_H(x) = \mathrm{rk}_G(x)$ for every $x \in \mathcal{U}(H)$.

Proof. Observe that the definition makes sense since every projection in $\mathcal{U}(G)$ already lies in $\mathcal{N}(G)$, as noted in Proposition 4.2.1(1.). Moreover, p is a positive operator, and if $(e)p = \sum_{g\in G} a_g g$ with $\sum_{g\in G} |a_g|^2 = C < \infty$, then

$$\mathrm{Tr}(p) = \langle (e)p, e\rangle = \langle (e)pp^*, e\rangle = \langle (e)p, (e)p\rangle = C.$$

Since $\langle (e)p, e\rangle = a_e$, we have that $a_e = C$ is positive and real with $a_e = \sum_{g\in G} |a_g|^2 \geq a_e^2$, and hence $\mathrm{Tr}(p) \in [0,1]$. Moreover, $\mathrm{Tr}(1_{\mathcal{U}(G)}) = \mathrm{Tr}(\mathrm{id}_{\ell^2(G)}) = \langle e, e\rangle = 1$, and since Tr is faithful, we have $0 = \mathrm{Tr}(p) = \mathrm{Tr}(pp^*)$ if and only if $p = 0$. This implies that $\mathrm{rk}_G(x) \in [0,1]$ for every $x \in \mathcal{U}(G)$, $\mathrm{rk}_G(1_{\mathcal{U}(G)}) = 1$ and $\mathrm{rk}_G(x) = 0$ if and only if $x = 0$. In particular, rk_G satisfies (PR1) of a pseudo-rank function and it is faithful.

Now, by Proposition 4.2.1(3.), for every $x \in \mathcal{U}(G)$ there exists a partial isometry $u \in \mathcal{N}(G)$ such that $\mathrm{RP}(x) = u^*u$ and $\mathrm{RP}(x^*) = uu^*$. Therefore, by the trace property one has

$$\mathrm{rk}_G(x) = \mathrm{Tr}(uu^*) = \mathrm{Tr}(u^*u) = \mathrm{rk}_G(x^*).$$

To show (PR2) note that if $x, y \in \mathcal{U}(G)$, then

$$\mathcal{U}(G)p = \mathcal{U}(G)xy \subseteq \mathcal{U}(G)y = \mathcal{U}(G)q,$$

where $p = \mathrm{RP}(xy)$ and $q = \mathrm{RP}(y)$. Thus $pq = p$ and, since both are projections, $p = p^* = q^*p^* = qp$. As a consequence,

$$(q-p)(q-p)^* = qq^* - qp^* - pq + pp^* = q - p - p + p = q - p,$$

i.e., $q - p$ is a projection, and hence by linearity of the trace,

$$\mathrm{rk}_G(y) - \mathrm{rk}_G(xy) = \mathrm{Tr}(q) - \mathrm{Tr}(p) = \mathrm{Tr}(q-p) \geq 0.$$

Since we have shown that $\mathrm{Tr}(\mathrm{RP}(x)) = \mathrm{Tr}(\mathrm{RP}(x^*)) = \mathrm{Tr}(\mathrm{LP}(x))$, we can argue similarly with the right ideals $xy\mathcal{U}(G) \subseteq x\mathcal{U}(G)$ to prove that $\mathrm{rk}_G(x) \geq \mathrm{rk}_G(xy)$, and hence $\mathrm{rk}_G(xy) \leq \min\{\mathrm{rk}_G(x), \mathrm{rk}_G(y)\}$.

Finally, let $e, f \in \mathcal{U}(G)$ be orthogonal idempotents in $\mathcal{U}(G)$. Set $p = \mathrm{RP}(e)$ and $q = \mathrm{RP}(f)$. Then using the properties in Proposition 4.1.5 and Corollary 4.1.6 of the projection lattice in a $*$-regular ring, we have by orthogonality of the idempotents

$$\mathcal{U}(G)(e+f) = \mathcal{U}(G)e + \mathcal{U}(G)f = \mathcal{U}(G)p + \mathcal{U}(G)q = \mathcal{U}(G)(q + \mathrm{RP}(p(1-q))).$$

We claim that $p = \mathrm{LP}(p(1-q))$. Indeed, note that $qf = q$ and $pe = p$. Therefore $p(1-q)e = p(1-qf)e = pe = p$ and consequently $p\mathcal{U}(G) \subseteq p(1-q)\mathcal{U}(G)$. Since the other containment is clear, the claim is proved and

$$\begin{aligned}\mathrm{rk}_G(e+f) &= \mathrm{Tr}(\mathrm{RP}(p(1-q))) + \mathrm{Tr}(q) = \mathrm{Tr}(\mathrm{LP}(p(1-q))) + \mathrm{Tr}(q)\\ &= \mathrm{Tr}(p) + \mathrm{Tr}(q) = \mathrm{rk}_G(e) + \mathrm{rk}_G(f).\end{aligned}$$

Therefore, rk_G is a faithful pseudo-rank function on $\mathcal{U}(G)$. If $H \leq G$, we saw that the embeddings $\mathcal{N}(H) \to \mathcal{N}(G)$ and $\mathcal{U}(H) \to \mathcal{U}(G)$ are compatible with traces and right projections RP, respectively. Consequently, for $x \in \mathcal{U}(H) \subseteq \mathcal{U}(G)$, we have by definition that

$$\mathrm{rk}_H(x) = \mathrm{Tr}_{\mathcal{N}(H)}(\mathrm{RP}(x)) = \mathrm{Tr}_{\mathcal{N}(G)}(\mathrm{RP}(x)) = \mathrm{rk}_G(x).$$

□

Recall that pseudo-rank functions can be extended uniquely to Sylvester matrix rank functions (see Proposition 1.3.9), and hence rk_G defines a faithful Sylvester matrix rank function on $\mathcal{U}(G)$ (cf. [JiLi21, Proposition 2.4]). Although we give here a slightly different proof, the following result is due to P. Linnell and T. Schick (see [Lin98, Lemma 12.3] and [Sch00**, Lemma 3.4], or [Lüc02, Lemma 10.39]).

Proposition 4.2.3. *Let G be a torsion-free group, K a subfield of $\mathbb{C}$. Then rk_G takes integer values on matrices over $K[G]$ if and only if the division closure of $K[G]$ in $\mathcal{U}(G)$ is a division ring.*

Proof. Let $\mathcal{D}$ and R denote, respectively, the division closure and the rational closure of $K[G]$ in $\mathcal{U}(G)$, and recall that $\mathcal{D} \subseteq R$ (see Remark 3.3.7).

If $\mathcal{D}$ is a division ring, then the restriction of rk_G to $\mathcal{D}$ coincides with its unique Sylvester matrix rank function, and hence takes integer values on matrices over $\mathcal{D}$, in particular over $K[G]$. Conversely, if a is a non-zero element of R, then by Proposition 3.3.8 there exists $r \geq 0$, P, Q invertible matrices over R and M a matrix over $K[G]$ such that

$$\begin{pmatrix} I_r & 0 \\ 0 & a \end{pmatrix} = PMQ$$

and hence, since rk_G is a rank on $\mathcal{U}(G)$, we must have

$$r + \mathrm{rk}_G(a) = \mathrm{rk}_G(I_r \oplus a) = \mathrm{rk}_G(PMQ) = \mathrm{rk}_G(M),$$

from where $\mathrm{rk}_G(a) \in \mathbb{Z}$. Since the rank of an element is at most one and rk_G is faithful, this implies that $\mathrm{rk}_G(a) = 1$, and hence by Lemma 1.3.12 a is invertible over $\mathcal{U}(G)$. Thus, $a^{-1} \in R$ (because R is division closed), what means that R is a division ring. This implies that $\mathcal{D}$ is also a division ring (in fact, $R = \mathcal{D}$), and the proof is finished. □

A torsion-free group satisfying any (and hence each) of the equivalent statements in Proposition 4.2.3 is said to satisfy the *strong Atiyah conjecture over K*. The general statement, for a non-necessarily torsion-free group G (with a bound on the orders of finite subgroups), is usually given by means of a dimension function $\dim_{\mathcal{U}(G)}$ defined for left $\mathcal{U}(G)$-modules. Let us introduce it here and explain its relation with rk_G.

In the first place ([Lüc02, Page 238]), if P is a finitely generated projective left $\mathcal{N}(G)$-module, we can choose an idempotent $n \times n$ matrix $A = (a_{ij})$ over $\mathcal{N}(G)$ such that the image of the map $r_A^{\mathcal{N}(G)} : \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ given by right multiplication by A is isomorphic to P, and the real number

$$\dim_{\mathcal{N}(G)}(P) = \mathrm{Tr}(A) := \sum_{i=1}^{n} \mathrm{Tr}(a_{ii})$$

is independent of the choice of A. Now ([Lüc02, Page 329, eq. 8.25]), if Q is a finitely generated projective $\mathcal{U}(G)$-module, there exists a finitely generated projective $\mathcal{N}(G)$-module P, unique up to isomorphism (see [Lüc02, Theorem 8.22(7) and (8)]) such that $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} P \cong Q$, and one sets

$$\dim_{\mathcal{U}(G)}(Q) = \dim_{\mathcal{N}(G)}(P).$$

This defines a (normalized) dimension function on $\mathcal{U}(G)$ (check particularly [Lüc02, Lemma 8.27] and note that $\dim_{\mathcal{U}(G)}(\mathcal{U}(G)) = \dim_{\mathcal{N}(G)}(\mathcal{N}(G)) = \mathrm{Tr}(\mathrm{id}_{\ell^2(G)}) = 1$), and a comprehensive list with the properties of $\dim_{\mathcal{U}(G)}$ can be found in [Rei98, Sections 3.2 & 3.3] or [Lüc02, Section 8.3]). In particular ([Lüc02, Theorems 6.7 & 8.29]), $\dim_{\mathcal{U}(G)}$ and $\dim_{\mathcal{N}(G)}$ can be extended to arbitrary modules, these extensions are additive on short exact sequences and satisfy, for every left $\mathcal{N}(G)$-module M,

$$\dim_{\mathcal{N}(G)}(M) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M).$$

The dimension function $\dim_{\mathcal{U}(G)}$ is in fact the dimension function associated to rk_G as defined above.

Lemma 4.2.4. rk_G *is the pseudo-rank function associated to* $\dim_{\mathcal{U}(G)}$.

Proof. Recall from Proposition 1.3.8 that the pseudo-rank function associated to $\dim_{\mathcal{U}(G)}$ is the unique one satisfying $\mathrm{rk}(x) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G)x)$ for every $x \in \mathcal{U}(G)$. Set $p = \mathrm{RP}(x)$ so that $\mathcal{U}(G)x = \mathcal{U}(G)p$. Then $p \in \mathcal{N}(G)$ and, since $\mathcal{U}(G)$ is a flat right $\mathcal{N}(G)$-module by Proposition 4.2.1(2) and Proposition 3.1.8 one has that (cf. [Rot09, Corollary 3.59])

$$\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \mathcal{N}(G)p \cong \mathcal{U}(G)p.$$

Moreover, since p is a projection, $\mathcal{N}(G) = \mathcal{N}(G)p \oplus \mathcal{N}(G)(1-p)$, so in particular $\mathcal{N}(G)p$ is a finitely generated projective left $\mathcal{N}(G)$-module. Moreover, $\mathcal{N}(G)p = \mathrm{im}\, r_p^{\mathcal{N}(G)}$, where $r_p^{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathcal{N}(G)$ denotes the homomorphism given by right multiplication by p, so by definition we have

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G)x) = \dim_{\mathcal{U}(G)}(\mathcal{U}(G)p) = \dim_{\mathcal{N}(G)}(\mathcal{N}(G)p) = \mathrm{Tr}(p) = \mathrm{rk}_G(x).$$

□

The strong Atiyah conjecture for a group G over a subfield K of $\mathbb{C}$ predicts the possible values that $\dim_{\mathcal{U}(G)}$ can take on the $\mathcal{U}(G)$-modules induced by finitely presented $K[G]$-modules, or equivalently in view of the previous lemma, the possible values that rk_G can take on matrices over $K[G]$. The following is detailed essentially in [Lüc02, Lemma 10.7], and will allow us to give equivalent reformulations of the strong Atiyah conjecture.

Let A be any $n \times m$ matrix over $K[G]$, and denote by $r_{A,K}$ (resp. $r_{A,\mathcal{N}}$ and $r_{A,\mathcal{U}}$) the map $K[G]^n \to K[G]^m$, (resp. with $\mathcal{N}(G)$ and $\mathcal{U}(G)$) given by right multiplication by A. By right exactness of the functor $\mathcal{N}(G) \otimes_{K[G]} \square$ one has $\operatorname{coker} r_{A,\mathcal{N}} \cong \mathcal{N}(G) \otimes_{K[G]} \operatorname{coker} r_{A,K}$ as $\mathcal{N}(G)$-modules and similarly $\operatorname{coker} r_{A,\mathcal{U}} \cong \mathcal{U}(G) \otimes_{K[G]} \operatorname{coker} r_{A,K}$ as $\mathcal{U}(G)$-modules. From additivity of $\dim_{\mathcal{N}(G)}$ and exactness of the short exact sequences

$$0 \to \ker r_{A,\mathcal{N}} \to \mathcal{N}(G)^n \xrightarrow{r_{A,\mathcal{N}}} \operatorname{im} r_{A,\mathcal{N}} \to 0$$

and

$$0 \to \operatorname{im} r_{A,\mathcal{N}} \to \mathcal{N}(G)^m \to \operatorname{coker} r_{A,\mathcal{N}} \to 0$$

we deduce the following list of equalities

$$\begin{aligned}
\dim_{\mathcal{N}(G)}(\ker r_{A,\mathcal{N}}) &= n - m + \dim_{\mathcal{N}(G)}(\operatorname{coker} r_{A,\mathcal{N}}) \\
&= n - m + \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{K[G]} \operatorname{coker} r_{A,K}) \\
&\overset{\triangle}{=} n - m + \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{N}[G]} \mathcal{N}(G) \otimes_{K[G]} \operatorname{coker} r_{A,K}) \\
&= n - m + \dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{K[G]} \operatorname{coker} r_{A,K}) \\
&= n - m + \dim_{\mathcal{U}(G)}(\operatorname{coker} r_{A,\mathcal{U}}) \\
&\overset{\blacktriangle}{=} n - m + (m - \operatorname{rk}_G(A)) = n - \operatorname{rk}_G(A),
\end{aligned}$$

where $\triangle$ comes from the relation between $\dim_{\mathcal{N}(G)}$ and $\dim_{\mathcal{U}(G)}$ and $\blacktriangle$ holds because rk_G and $\dim_{\mathcal{U}(G)}$ are associated.

There is also a well-defined related notion of dimension for closed left G-invariant subspaces of $\ell^2(G)^n$, the so-called *von Neumann dimension* (cf. [Lüc02, Definitions 1.8 and 1.10]). If V is such a subspace, we can consider the projection $p_V : \ell^2(G)^n \to \ell^2(G)^n$ onto V and define

$$\dim_{\mathcal{N}(G)}(V) = \sum_{i=1}^{n} \langle (e_i)p_V, e_i \rangle_{\ell^2(G)^n}$$

where $e_i \in \ell^2(G)^n$ is the element with $e \in G$ in the i-th position and zeros everywhere else. If in the previous setting we denote by $r_{A,\ell^2} : \ell^2(G)^n \to \ell^2(G)^m$ the map induced by right multiplication by A, then it can be shown ([Lüc02, Lemma 10.7 & Theorem 6.24]) that

$$\dim_{\mathcal{N}(G)}(\ker r_{A,\ell^2}) = \dim_{\mathcal{N}(G)}(\ker r_{A,\mathcal{N}})$$

where on the left we are using the dimension for subspaces and on the right we are using the dimension for $\mathcal{N}(G)$-modules. Therefore, with the previous discussion and Proposition 4.2.3 we have shown the following.

Proposition 4.2.5. *Let K be a subfield of $\mathbb{C}$, let G be a group and assume that* $\mathrm{lcm}(G)$*, the least common multiple of the orders of its finite subgroups, is finite. Then, the following are equivalent:*

1. *For every matrix A over $K[G]$,*

$$\dim_{\mathcal{N}(G)}(\ker r_{A,\ell^2}) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z},$$

2. *For every finitely presented left $K[G]$-module M,*

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{K[G]} M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

3. *For every finitely presented left $K[G]$-module M,*

$$\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{K[G]} M) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

4. *For every matrix A over $K[G]$,*

$$\mathrm{rk}_G(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

If G is torsion-free, then $\mathrm{lcm}(G) = 1$ *and these are equivalent to*

5. *The division closure of $K[G]$ in $\mathcal{U}(G)$ is a division ring.*

Definition 4.2.6. Let G be a group with an upper bound on the orders of finite subgroups. If G satisfies any (and hence) each of the equivalent statements in Proposition 4.2.5 over the subfield K of $\mathbb{C}$, then we say that G satisfies the *strong Atiyah conjecture over K.*

If G is not assumed to have an upper bound for the order of its finite subgroups, then the numbers appearing in Proposition 4.2.5 may even fail to be rational (see, for instance, [Aus13]).

To finish the section, let H be a torsion-free group and let $N \lhd H$ be a normal subgroup of H. Since every $h \in H$ normalizes N (in the sense that $hNh^{-1} = N$ in H), it can be shown that h also normalizes $\mathbb{C}[N], \mathcal{N}(N)$ and $\mathcal{U}(N)$ (more generally, every automorphism of N extends to automorphisms of $\mathbb{C}[N], \mathcal{N}(N)$ and $\mathcal{U}(N)$). In particular, we have that $h\mathcal{U}(N)h^{-1} = \mathcal{U}(N)$ in $\mathcal{U}(H)$. Moreover, elements in different cosets of N in H can be shown to be right (and hence left since $h\mathcal{U}(N)h^{-1} = \mathcal{U}(N)$ for every h) $\mathcal{U}(N)$-linearly independent (cf. the discussion after Problem 4.5 in [Lin06]). Accordingly, the crossed product structure $\mathbb{C}[H] = \mathbb{C}[N] * H/N$ extends to a crossed product structure $\mathcal{U}(N) * H/N$ that can be realized inside $\mathcal{U}(H)$. In other words, one has (cf. [Lüc02, 10.57(i)])

$$\mathbb{C}[H] = \mathbb{C}[N] * H/N \subseteq \mathcal{U}(N) * H/N \subseteq \mathcal{U}(H).$$

Let T be a transversal of N in H containing $e \in H$. As we said, the elements in T are right and left $\mathcal{U}(N)$-linearly independent. Moreover, if $\mathcal{D}_{\mathbb{C}[N]}$ is the division closure of $\mathbb{C}[N]$ in $\mathcal{U}(H)$, then the subring $\mathcal{D}_{\mathbb{C}[N]}H$ of $\mathcal{D}_{\mathbb{C}[H]}$ generated by $\mathcal{D}_{\mathbb{C}[N]}$ and H can also be given a crossed product structure. More precisely, one can show that $t\mathcal{D}_{\mathbb{C}[N]}t^{-1} = \mathcal{D}_{\mathbb{C}[N]}$ for every $t \in T$ and hence $\mathcal{D}_{\mathbb{C}[N]}H = \sum_{t\in T} \mathcal{D}_{\mathbb{C}[N]}t$, being the sum direct because $\sum_{t\in T} \mathcal{U}(N)t$ is direct (cf. [Lin98, Lemmas 9.2 & 9.3]).

The same holds for the $*$-regular closure $\mathcal{R}_{\mathbb{C}[N]}$. For this, take any $t \in T$, set $\mathcal{R}_0 = \mathbb{C}[N]$ and observe that since N is normal in H, $t\mathcal{R}_0t^{-1} = \mathcal{R}_0$. Now assume that we have seen that $t\mathcal{R}_nt^{-1} = \mathcal{R}_n$ for some $n \geq 0$, where $\mathcal{R}_n$ is the n-th step in the inductive construction of $\mathcal{R}_{\mathbb{C}[N]}$. We need to show that $t\mathcal{R}_{n+1}t^{-1} = \mathcal{R}_{n+1}$. Indeed, let $y \in \mathcal{R}_{n+1}$ be such that $y = x^{[-1]}$, $x \in \mathcal{R}_n$.

We claim that $tyt^{-1} = (txt^{-1})^{[-1]}$, and hence by construction of $\mathcal{R}_{n+1}$ and the induction hypothesis, $tyt^{-1} \in \mathcal{R}_{n+1}$. According to Remark 4.1.3 we just need to show that, if $a = tyt^{-1}$ and $b = txt^{-1}$, then both ab and ba are projections such that $aba = a$, $bab = b$. Since the involution $*$ acts on an element $h \in H$ as $h^* = h^{-1}$, we see that

$$ab(ab)^* = tyx(yx)^*t^{-1} = tyxt^{-1} = ab,$$

where we have used that yx is a projection. Therefore, ab, and similarly ba, are projections. The claim follows because $aba = tyxyt^{-1} = tyt^{-1} = a$ and analogously $bab = b$, and hence we have seen that for every generator a of $\mathcal{R}_{n+1}$, $tat^{-1} \in \mathcal{R}_{n+1}$. Since any other element in $\mathcal{R}_{n+1}$ is obtained from these generators by addition, substraction and products, we conclude that $t\mathcal{R}_{n+1}t^{-1} \subseteq \mathcal{R}_{n+1}$. Proceeding analogously, $t^{-1}\mathcal{R}_{n+1}t \subseteq \mathcal{R}_{n+1}$ and hence we have equality.

Consequently, $t\mathcal{R}_{\mathbb{C}[N]}t^{-1} = \mathcal{R}_{\mathbb{C}[N]}$, and hence, $\mathcal{R}_{\mathbb{C}[N]}H = \sum_{t\in T} \mathcal{R}_{\mathbb{C}[N]}t$. The same reason given before shows that actually $\mathcal{R}_{\mathbb{C}[N]}H = \bigoplus_{t\in T} \mathcal{R}_{\mathbb{C}[N]}t = \mathcal{R}_{\mathbb{C}[N]} * H/N$ inherits the crossed product structure coming from $\mathcal{U}(N) * H/N$.

If H/N is infinite cyclic and $t \in H$ is such that $H/N = \langle Nt \rangle$, we can take T to be the set consisting of the powers of t, and the previous discussion shows that these powers are left and right linearly independent over $\mathcal{R}_{\mathbb{C}[N]}$ (resp. $\mathcal{D}_{\mathbb{C}[N]}$, $\mathcal{U}(N)$). Hence, we can realize the given crossed product as $\mathcal{R}_{\mathbb{C}[N]} * H/N \cong \mathcal{R}_{\mathbb{C}[N]}[x^{\pm 1}; \tau]$, where $t \mapsto x$ and τ is given by left conjugation by t, and the same holds true for $\mathbb{C}[N]$, $\mathcal{D}_{\mathbb{C}[N]}$ and $\mathcal{U}(N)$. In particular, if G is a locally indicable group, all of this implies that $\mathcal{D}_{\mathbb{C}[G]}$ is the natural candidate to be the Hughes-free division $\mathbb{C}[G]$-ring of fractions.

In fact, under these hypothesis, A. Jaikin-Zapirain proved in [Jai19S, Corollary 12.2] that the rank rk_H, as a Sylvester matrix rank function on $\mathcal{R}_{\mathbb{C}[N]} * H/N$, is the "natural extension" of the rank rk_N on $\mathcal{R}_{\mathbb{C}[N]}$. Fixing the transversal T, i.e., fixing the crossed product basis $u_{Nh} = t^k$, where $t^k \in T$ is the unique one with $Nh = Nt^k$, and fixing the Følner basis of H/N given by $F_n = \{N, Nt, \ldots, Nt^{n-1}\}$, we can see that his definition [Jai19S, Theorem 8.2] of natural extension of an exact Sylvester module rank function coincides with ours as shown through Proposition 1.5.1 and Proposition 1.5.4.

As mentioned at the end of Section 4.1, the automorphisms of $\mathbb{C}[N]$ and $\mathcal{R}_{\mathbb{C}[N]}$ induced by left conjugation by t, which we denote identically by τ, are actually $*$-automorphisms, and rk_N is τ-compatible both as a rank on $\mathbb{C}[N]$ and $\mathcal{R}_{\mathbb{C}[N]}$ since it is the restriction of

rk_H. Note that $(\mathcal{R}_{\mathbb{C}[N]}, \mathrm{rk}_N)$ is the $*$-regular envelope of rk_N as a rank on $\mathbb{C}[N]$ and that the diagrams

$$\begin{tikzcd} \mathbb{C}[N] \arrow[r] \arrow[d, "\tau"'] & \mathcal{R}_{\mathbb{C}[N]} \arrow[d, "\tau"] \\ \mathbb{C}[N] \arrow[r] & \mathcal{R}_{\mathbb{C}[N]} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}[N] * H/N \arrow[r] \arrow[d, "\cong"'] & \mathcal{R}_{\mathbb{C}[N]} * H/N \arrow[d, "\cong"] \\ \mathbb{C}[N][x^{\pm 1}; \tau] \arrow[r] & \mathcal{R}_{\mathbb{C}[N]}[x^{\pm 1}; \tau] \end{tikzcd}$$

commute. Thus, Proposition 4.1.28 tells us that the natural extension of the rank function $\mathrm{rk}_N \in \mathbb{P}(\mathbb{C}[N])$ to $\mathbb{C}[H] = \mathbb{C}[N] * H/N \cong \mathbb{C}[N][x^{\pm 1}; \tau]$ coincides precisely with $\mathrm{rk}_H \in \mathbb{P}(\mathbb{C}[H])$, because this is by [Jai19S, Corollary 12.2] the restriction to $\mathbb{C}[H]$ of the natural extension of $\mathrm{rk}_N \in \mathbb{P}(\mathcal{R}_{\mathbb{C}[N]})$. Gluing these arguments, and in the language of Section 3.4, we have the following result.

Proposition 4.2.7. *Let H be a group, $N \lhd H$ a normal subgroup such that H/N is infinite cyclic. Then rk_H, as a rank function on $\mathbb{C}[H]$, is the natural extension of rk_N as a rank function on $\mathbb{C}[N]$. Therefore, if G is a locally indicable group, then rk_G, as a rank function on $\mathbb{C}[G]$, is Hughes-free.*

4.3 Rational semirings and the notion of complexity

In this section we introduce rational U-semirings and the notion of complexity that shall be used for the inductive proof of the strong Atiyah conjecture for locally indicable groups in Section 4.4. Most of the theory presented here was developed in [DHS04] in order to give an alternative proof of the theorem of Hughes about the uniqueness of the Hughes-free division ring of fractions (see Theorem 3.4.23) and it is completely detailed in [Sán08], which we stick to as our main reference.

Let us start with the definition of rational U-semiring and morphisms between them ([Sán08, Definitions 1.35 & Definitions 1.42]).

Definition 4.3.1. Let U be a multiplicative group.

- A *semiring* is a set R together with an addition $+$, which makes it an additive (commutative) semigroup, and a product $\cdot$, which makes it a multiplicative monoid (in particular it has an identity element 1_R), and which is left and right distributive over the addition. A *morphism of semirings* is a map $\Phi : R_1 \to R_2$ satisfying, for all $r, r' \in R$:

$$\Phi(r + r') = \Phi(r) + \Phi(r'),$$
$$\Phi(rr') = \Phi(r)\Phi(r'),$$
$$\Phi(1_{R_1}) = 1_{R_2}.$$

- A *left U-set* is a set X with a map $U \times X \to X$, $(u, x) \mapsto ux$ such that $1_U x = x$ and $u(vx) = (uv)x$ for all $u, v \in U, x \in X$. *Right U-sets* are defined analogously, and a *U-biset* is a set which is both a left and right U-set and such that, for all

$u, v \in U$, $x \in X$, $(ux)v = u(xv)$. A *morphism of U-bisets* is a map $\Phi : X_1 \to X_2$ satisfying, for all $u, v \in U, x \in X$,

$$\Phi(uxv) = u\Phi(x)v.$$

- A *U-semiring* is a semiring R together with a morphism of monoids $\phi : U \to R$, i.e., a map satisfying, for every $u, v \in U$,

$$\phi(uv) = \phi(u)\phi(v), \qquad \phi(1_U) = 1_R.$$

  In particular, R has a U-biset structure given by the products in R $ur := \phi(u)r$ and $ru := r\phi(u)$. A *morphism of U-semirings* is a map $\Phi : R_1 \to R_2$ which is both a morphism of semirings and of U-bisets.

- A *rational semiring* is a semiring R endowed with a map $\diamond : R \to R$, $r \mapsto r^\diamond$, and a *rational U-semiring* is a U-semiring R with a $\diamond$-map satisfying, for all $u, v \in U$, $r \in R$,

$$(urv)^\diamond = v^{-1}r^\diamond u^{-1}.$$

  A *morphism of rational U-semirings* is a morphism of U-semirings $\Phi : R_1 \to R_2$ with the additional property that, for all $r \in R_1$

$$\Phi(r^\diamond) = \Phi(r)^\diamond.$$

Note that if V is a subgroup of U, then every rational U-semiring is naturally a rational V-semiring ([Sán08, Remark 1.44]). Each of the following subsections is devoted to introduce a particular example of rational U-semiring.

4.3.1 Finite rooted trees

This example corresponds to [Sán08, Section 5.2]. Let $\mathcal{T}$ be the set of all isomorphisms classes of finite (oriented) rooted trees. We will just recall here that $\mathcal{T}$ has a well-order satisfying some desirable properties and that can be trivially seen to be a U-semiring for any multiplicative group U. This order will define a measure of complexity of elements in the other examples of rational U-semirings that we introduce later.

Before defining the operations in $\mathcal{T}$, let us introduce some related notions and notation ([Sán08, Definitions 5.7]). Let $0_\mathcal{T}$ be the one-vertex tree and $1_\mathcal{T}$ be the one-edge rooted tree.

Definition 4.3.2. Let $X \in \mathcal{T}$.

- We denote by $\mathrm{fam}(X)$ the finite family of finite rooted trees (with multiplicity) obtained from X by deleting the root and all incident edges. The root of every element in $\mathrm{fam}(X)$ is the one incident to the deleted edge. If $X = 0_\mathcal{T}$, then $\mathrm{fam}(X)$ is the empty set.

- The *width* of X is the number of elements in $\mathrm{fam}(X)$.

- The *height* of X is defined recursively. We set $\operatorname{height}(0_{\mathcal{T}}) = 0$ and then, for $X \neq 0_{\mathcal{T}}$, we set $\operatorname{height}(X)$ as the maximum height of the elements in $\operatorname{fam}(X)$ plus one.

- We denote by $\exp(X)$ the tree obtained from X by adding a new vertex, which is declared to be the root of $\exp(X)$, and a new edge joining it to the root of X. In this case $\operatorname{height}(\exp(X)) = \operatorname{height}(X) + 1$ and $\operatorname{fam}(\exp(X)) = \{X\}$. Note that $1_{\mathcal{T}} = \exp(0_{\mathcal{T}})$.

Now we can define the operations in $\mathcal{T}$ that makes it a rational semiring, so take $X, Y \in \mathcal{T}$.

Sum The sum $X + Y$ consists of identifying the roots of X and Y, and declaring it to be the root of the resulting tree. With this operation $\mathcal{T}$ is an additive monoid with neutral element $0_{\mathcal{T}}$. Note that $X + Y = 0_{\mathcal{T}}$ if and only if $X = Y = 0_{\mathcal{T}}$.

Product The product $X \cdot Y$ consists of adding pairwise the elements of $\operatorname{fam}(X)$ with the elements of $\operatorname{fam}(Y)$, and then connecting all the resulting finite rooted trees by adding a new vertex (the root of $X \cdot Y$) with incident edges to their roots. In other words,

$$X \cdot Y = \sum_{\substack{X' \in \operatorname{fam}(X) \\ Y' \in \operatorname{fam}(Y)}} \exp(X' + Y').$$

With this operation, $\mathcal{T}$ is a commutative multiplicative monoid with identity element $1_{\mathcal{T}}$. The product can be shown to be distributive over the sum and hence $\mathcal{T}$ is a semiring in which $0_{\mathcal{T}} \cdot X = X \cdot 0_{\mathcal{T}} = 0_{\mathcal{T}}$ for every $X \in \mathcal{T}$ (cf. [Sán08, Lemma 5.8]). Note that $X \cdot Y = 1_{\mathcal{T}}$ if and only if $X = Y = 1_{\mathcal{T}}$.

$\diamond$-map The rational operation is given by $X^{\diamond} = \exp^2(X)$.

With these operations, $\mathcal{T}$ is now a rational semiring. Moreover, if U is any multiplicative group, then $\mathcal{T}$ is a rational U-semiring by means of the trivial map $\phi : U \to \mathcal{T}$ with $\phi(u) = 1_{\mathcal{T}}$ for every $u \in U$.

As mentioned above, one important property is that $\mathcal{T}$ is well-ordered. Here, we choose to follow [Sán08] for the sake of homogeneity, because we refer to it for most of the properties and claims. We have the following ([Sán08, Definition 5.11 & Lemma 5.15]).

Lemma 4.3.3. *Set $\mathcal{T}_{n,m}$, $n, m \in \mathbb{N}$, to be the subset of $\mathcal{T}$ consisting of all elements with* $\operatorname{height}(X) \leq n - 1$ *(and any* width*), or (*$\operatorname{height}(X) = n$ *and* $\operatorname{width}(X) \leq m$*). Then $\mathcal{T}_{1,0} = \{0_{\mathcal{T}}\}$ and we let $0_{\mathcal{T}}$ be the least element of $\mathcal{T}$.*

Suppose that we have ordered $\mathcal{T}_{n,0}$ for some $n \geq 1$, and assume that we have ordered $\mathcal{T}_{n,m-1}$ for some $m \geq 1$. Now take any nonzero $X, Y \in \mathcal{T}_{n,m}$.

- *Since the elements in* $\operatorname{fam}(X)$ *belong to $\mathcal{T}_{n,0}$, we can define* $\log(X)$ *to be the largest element in* $\operatorname{fam}(X)$*.*

- *Then* $\exp(\log(X))$ *is a summand of* X*, and hence it has a unique complement* $X - \exp(\log(X))$*, i.e., such that*

$$X = \exp(\log(X)) + (X - \exp(\log(X))).$$

- *Now* $\log(X)$ *belongs to* $\mathcal{T}_{n,0}$ *(we reduced the* height*),* $X - \exp(\log(X))$ *belongs to* $\mathcal{T}_{n,m-1}$ *(we reduced the* width*), and we have the corresponding result for* Y*, so we can compare them.*

- *We declare* $X > Y$ *if either* $\log(X) > \log(Y)$ *or* ($\log(X) = \log(Y)$ *and* $X - \exp(\log(X)) > Y - \exp(\log(Y))$). *We have ordered* $\mathcal{T}_{n,m}$.

By induction on m*, we have ordered* $\mathcal{T}_{n,m}$ *for every* m*, and hence* $\mathcal{T}_{n+1,0}$*. By induction on* n*, we have ordered* $\mathcal{T}$*, and this is a well order.*

In particular, if height$(X) >$ height(Y), then $X > Y$ (see [Sán08, Remark 5.14]). This order satisfies, among many other properties, the following (cf. [Sán08, Remark 5.18]).

Lemma 4.3.4. *Let* $X, Y, X', Y' \in \mathcal{T}$.

(i) If $X' \leq X$ *and* $Y' \leq Y$*, then* $X' + Y' \leq X + Y$*, and equality holds if and only if* $X' = X$ *and* $Y' = Y$*. In particular, if* $Y \neq 0_{\mathcal{T}}$*, then* $X < X + Y$*.*

(ii) If $X' \leq X$ *and* $Y' \leq Y$*, then* $X' \cdot Y' \leq X \cdot Y$*. If* $X', Y' \neq 0_{\mathcal{T}}$*, then equality holds if and only if* $X' = X$ *and* $Y' = Y$*. In particular, if* $X, Y \neq 0_{\mathcal{T}}$*, then* $X \leq X \cdot Y$ *and they are equal if and only if* $Y = 1_{\mathcal{T}}$*.*

4.3.2 The universal rational U-semiring

Given a multiplicative group U, the universal rational U-semiring $\mathrm{Rat}(U)$ is constructed inductively as a formal analog of the construction of a division or a $*$-regular closure, starting with the elements of U, constructing at each inductive step a bigger rational U-semiring by means of sums, products and rational operations $\diamond$ of the object in the previous step, and then taking unions. As in the previous example, before defining $\mathrm{Rat}(U)$, we present some definitions and notation.

- If X is a set, then the *free additive monoid* (or free commutative monoid) *on* X is $\mathbb{N}[X]$, the set of formal sums $\sum_{x\in X} n_x x$ ($n_x \in \mathbb{N}$) with finite support and endowed with the natural addition. The neutral element 0 is the formal sum with empty support, and the element $1x$ is written x. In this way we identify $X \subseteq \mathbb{N}[X]$.

  If X is a multiplicative monoid, then $\mathbb{N}[X]$ is a semiring with multiplicative monoid structure given by linearly extending the product in X, i.e.,

$$\left(\sum_{x\in X} n_x x\right)\left(\sum_{x\in X} m_x x\right) = \sum_{x\in X}\left(\sum_{yz=x} n_y m_z\right) x.$$

The identity element is then 1_X, and in this case X is a multiplicative submonoid of $\mathbb{N}[X]$. If additionally there is a morphism $U \to X$ of monoids, then $\mathbb{N}[X]$ is a U-semiring with the morphism of monoids given by the composition $U \to X \hookrightarrow \mathbb{N}[X]$.

The *free additive semigroup* on X is $\mathbb{N}[X]\backslash\{0\}$. If X is a multiplicative monoid, the sum and product of non-zero elements in $\mathbb{N}[X]$ is non-zero, and hence $\mathbb{N}[X]\backslash\{0\}$ is again a semiring (resp. U-semiring) with the previous operations.

In this way we shall construct *formal sums* of elements in the universal object that we are about to construct.

- If X is a U-biset, then we define an equivalence relation on $X \times X$ by $(x_1, x_2) \sim (x_1', x_2')$ if and only if there exists $u \in U$ such that $x_1 u = x_1'$ and $u^{-1}x_2 = x_2'$. Let $X \times_U X$ be the set of equivalence classes, where x_1x_2 denotes the equivalence class of (x_1, x_2). Hence, in $X \times_U X$ we have the equality $(xu)x' = x(ux')$, and a U-biset structure given by

$$u(x_1x_2) = (ux_1)x_2 \quad \text{and} \quad (x_1x_2)u = x_1(x_2u).$$

The map $(X \times_U X) \times_U X \to X \times_U (X \times_U X)$ given by $(x_1x_2)x_3 \mapsto x_1(x_2x_3)$ is an isomorphism of U-bisets ([Sán08, Lemma 5.25]), and the same holds for any finite number of copies of X and any placement of the parenthesis ([Sán08, Definition 5.26](a)). We define inductively $X^{\times_U^0} = U$, $X^{\times_U^1} = X$ and, for $n \geq 2$,

$$X^{\times_U^n} = X^{\times_U^{n-1}} \times_U X,$$

and we write $x_1 \dots x_n$ to mean $(x_1 \dots x_{n-1})x_n$. As above, $X^{\times_U^n}$ is a U-biset with

$$u(x_1x_2 \dots x_n) = (ux_1)x_2 \dots x_n \text{ and } (x_1x_2 \dots x_n)u = x_1x_2 \dots (x_nu).$$

We define now the U-biset $U\natural X$ as the disjoint union

$$U\natural X = \bigsqcup_{n=0}^{\infty} X^{\times_U^n}$$

One can show that the natural operation

$$(x_1 \cdots x_n) \cdot (y_1 \cdots y_n) = x_1 \cdots x_n y_1 \cdots y_n$$

makes $U\natural X$ a multiplicative monoid with identity element 1_U and that contains U as a submonoid ([Sán08, Definition 5.26]). We call $U\natural X$ the *free multiplicative U-monoid on X over U*. Somehow, we can think of it as the set of words in X of any length together with concatenation and modulo the relations coming from the multiplication in U and the U-biset structure of X.

In this way we shall construct *formal products* of elements of X. In addition, observe by the previous definitions that $\mathbb{N}[U\natural X]$ and $\mathbb{N}[U\natural X]\backslash\{0\}$ are both U-semirings.

- If X is a U-biset, then $X^\diamond$ denotes a disjoint copy of X together with a bijective map $X \to X^\diamond$, $x \mapsto x^\diamond$, and a U-biset structure given by $ux^\diamond v := (v^{-1}xu^{-1})^\diamond$ for all $u, v \in U$, $x \in X$. This will allow us to define a *formal rational operation* in X.

We can now define the next example of rational U-semiring ([Sán08, Definition 5.32]).

Definition 4.3.5. Let U be a multiplicative group. The *universal rational U-semiring* $\mathrm{Rat}(U)$ is defined as follows.

1. Consider the U-semiring $\mathbb{N}[U]\setminus\{0\}$, and set $X_0 = \emptyset$, $X_1 = (\mathbb{N}[U]\setminus\{0\})^\diamond$. Then X_0 is a U-sub-biset of X_1.

2. Suppose that $n \geq 1$, X_n is a U-biset and X_{n-1} a U-sub-biset of X_n. Consider the U-semiring $\mathbb{N}[U\natural X_n]$. Since $\mathbb{N}[U\natural X_{n-1}]$ is a U-sub-biset of $\mathbb{N}[U\natural X_n]$, then we have that $\mathbb{N}[U\natural X_n]\setminus\mathbb{N}[U\natural X_{n-1}]$ is a U-sub-biset, and we define

$$X_{n+1} = (\mathbb{N}[U\natural X_n]\setminus\mathbb{N}[U\natural X_{n-1}])^\diamond \cup X_n.$$

3. Then, $X = \bigcup X_n$ is a U-biset and

$$\mathrm{Rat}(U) := \mathbb{N}[U\natural X]\setminus\{0\}.$$

The $\diamond$-map can be shown to carry $\mathbb{N}[U]\setminus\{0\}$ to X_1, $\mathbb{N}[U\natural X_n]\setminus\mathbb{N}[U\natural X_{n-1}]$ to $X_{n+1}\setminus X_n$ for $n \geq 1$. Moreover, one can prove inductively that it carries bijectively $\mathbb{N}[U\natural X_n]\setminus\{0\}$ to X_{n+1} for $n \geq 0$, and $\mathrm{Rat}(U)$ to X [Sán08, Remark 5.33]. The U-semiring $\mathrm{Rat}(U) \cup \{0\}$ (where we add an absorbing zero) is (isomorphic to) the U-semiring $\mathbb{N}[U\natural X]$.

Observe that, starting from U, we construct $\mathrm{Rat}(U)$ by allowing at each step formal sums and products of the elements in the previous step, and then defining a formal rational operation on the new elements obtained this way. The universality of $\mathrm{Rat}(U)$ comes from the following property ([Sán08, Lemma 5.34]). We reproduce the proof to show how the inductive construction of $\mathrm{Rat}(U)$ is used.

Lemma 4.3.6. *If U is a multiplicative group and R a rational U-semiring, there exists a unique morphism of rational U-semirings $\Phi : \mathrm{Rat}(U) \to R$.*

If R has a zero element 0_R, then Φ extends to a morphism of additive (commutative) monoids $\Phi' : \mathrm{Rat}(U) \cup \{0\} \to R$, and if the zero is absorbing (i.e., $0_R r = r 0_R = 0_R$ for all $r \in R$), Φ' is a morphism of U-semirings.

Proof. Let $\phi : U \to R$ be the morphism of monoids defining the U-semiring structure of R, and set $\phi_0 : X_0 \to R$ to be the inclusion map ($X_0 = \emptyset$), which is a morphism of U-bisets.

Assume that we have defined a morphism of U-bisets $\phi_n : X_n \to R$ for some $n \geq 0$. This defines a unique morphism of U-semirings

$$\phi_n : \mathbb{N}[U\natural X_n]\setminus\{0\} \to R.$$

For an element $u \in U$, it is defined by $\phi_n(u) = \phi(u)$, for an element $x = x_1 \dots x_n$ of $U\natural X_n\setminus U$, it is defined by $\phi_n(x) = \phi_n(x_1)\dots\phi_n(x_n)$, and for an element $\sum n_x x \in \mathbb{N}[U\natural X_n]\setminus\{0\}$ it is given by $\phi_n(\sum n_x x) = \sum n_x \phi_n(x)$ (see [Sán08, Definition 5.24(c) &

Lemma 5.27]). Now, since the $\diamond$-map carries bijectively $\mathbb{N}[U\natural X_n]\backslash\{0\}$ to X_{n+1}, every element $y \in X_{n+1}$ is of the form $y = z^\diamond$ for some $z \in \mathbb{N}[U\natural X_n]\backslash\{0\}$, and hence the map

$$\phi_{n+1} : X_{n+1} \to R$$

given by $\phi_{n+1}(z^\diamond) = (\phi_n(z))^\diamond$ defines a morphism of U-bisets.

We have inductively defined ϕ_n for every n, and we need to show that ϕ_{n+1} coincides with ϕ_n over X_n. For the base of induction, it is clear that ϕ_1 coincides with ϕ_0 on X_0. Assume that we have shown that ϕ_n agrees with ϕ_{n-1} on X_{n-1}, and take an element $y \in X_n$. As before, $y = z^\diamond$ for some $z \in \mathbb{N}[U\natural X_{n-1}]\backslash\{0\}$, and hence, by definition of ϕ_{n+1}, the induction hypothesis and the way ϕ_n is extended to $\mathbb{N}[U\natural X_n]\backslash\{0\}$,

$$\phi_{n+1}(y) = \phi_{n+1}(z^\diamond) = (\phi_n(z))^\diamond = (\phi_{n-1}(z))^\diamond = \phi_n(z^\diamond) = \phi_n(y).$$

Therefore, the map $\Phi : X \to R$ that coincides with ϕ_n on X_n defines a morphism of U-bisets that induces as before a morphism of U-semirings $\Phi : \mathrm{Rat}(U) \to R$ that preserves the $\diamond$ operation. This map is uniquely determined by the map $\phi : U \to R$ with which we start, by the inductive construction of $\mathrm{Rat}(U)$.

For the last statement, one just need to define $\Phi'(0) = 0_R$. $\square$

As shown through [Sán08, Example 5.35], if V is a subgroup of U then the universal morphism given by the previous lemma

$$\Psi_{V,U} : \mathrm{Rat}(V) \to \mathrm{Rat}(U)$$

is naturally injective at every inductive step. More precisely, let X_n and Y_n denote, respectively, the U-bisets needed to construct $\mathrm{Rat}(V)$ and $\mathrm{Rat}(U)$, and let $\phi_n : X_n \to \mathrm{Rat}(U)$ be the map constructed in Lemma 4.3.6 to define $\Psi_{V,U}$ (i.e., ϕ_n is the restriction of $\Psi_{V,U}$ to X_n). Then ϕ_n is injective and $\phi_n(X_n) \subseteq Y_n$ is an *admissible* V-sub-biset of Y_n, meaning that $\phi_n(X_n)$ is closed under left and right multiplication by elements of V, and that for every $u \in U\backslash V$, we have $\phi_n(X_n) \cap \phi_n(X_n)u = \phi_n(X_n) \cap u\phi_n(X_n) = \emptyset$. Hence $\Psi_{V,U}$ itself is injective and $\mathrm{Rat}(V)$ can be identified with an admissible V-sub-biset of $\mathrm{Rat}(U)$, in particular a rational subsemiring of $\mathrm{Rat}(U)$.

An important consequence of Lemma 4.3.6 is that for every U, we obtain a morphism of U-semirings:

$$\mathrm{Tree} : \mathrm{Rat}(U) \cup \{0\} \to \mathcal{T}.$$

which is a morphism of rational U-semirings when restricted to $\mathrm{Rat}(U)$. The image $\mathrm{Tree}(\alpha)$ of $\alpha \in \mathrm{Rat}(U)$ is called its *complexity*. Because of the universal property of Rat, if V is a subgroup of U, we obtain a diagram of morphisms of rational V-semirings

$$\begin{tikzcd}
\mathrm{Rat}(U) \arrow[r, "\mathrm{Tree}_U"] & \mathcal{T} \\
\mathrm{Rat}(V) \arrow[u, hook, "\Psi_{V,U}"] \arrow[ur, "\mathrm{Tree}_V"'] &
\end{tikzcd}$$

Since the composition $\mathrm{Tree}_U \circ \Psi_{V,U}$ is a morphism of rational V-semirings, then the uniqueness of Tree_V implies that the diagram is commutative, and since $\Psi_{V,U}$ is injective, we can think of Tree_V as the restriction of Tree_U (cf. [Sán08, Example 5.36]). Under the identification $\mathrm{Rat}(V) \subseteq \mathrm{Rat}(U)$, this means that for $\alpha \in \mathrm{Rat}(V)$, $\mathrm{Tree}(\alpha)$ does not depend on whether we consider α as an element of $\mathrm{Rat}(V)$ or $\mathrm{Rat}(U)$.

The following lemma collects some of the properties of the complexity. In order to state it properly, add to $\mathcal{T}$ a new least element $\{-\infty\}$ and turn $\mathcal{T} \cup \{-\infty\}$ into a semiring by setting $\mathcal{T} + \{-\infty\} = \mathcal{T} \cdot \{-\infty\} = \{-\infty\} \cdot \mathcal{T} = \{-\infty\}$. Now define $\log(0_{\mathcal{T}}) = -\infty$ and $\log(-\infty) = -\infty$.

Lemma 4.3.7. *If $\alpha, \beta \in \mathrm{Rat}(U) \cup \{0\}$, then the following hold.*

(i) $\mathrm{Tree}(\alpha) = 0_{\mathcal{T}}$ *if and only if* $\alpha = 0$.

(ii) $\mathrm{Tree}(\alpha) = 1_{\mathcal{T}}$ *if and only if* $\alpha \in U$.

(iii) $\mathrm{Tree}(\alpha + \beta) = \mathrm{Tree}(\alpha) + \mathrm{Tree}(\beta)$.

(iv) $\mathrm{Tree}(\alpha) \leq \mathrm{Tree}(\alpha + \beta)$ *and they are equal if and only if* $\beta = 0$.

(v) $\mathrm{Tree}(\alpha\beta) = \mathrm{Tree}(\alpha)\mathrm{Tree}(\beta)$.

(vi) If $\alpha, \beta \neq 0$, *then* $\mathrm{Tree}(\alpha) \leq \mathrm{Tree}(\alpha\beta)$ *and they are equal if and only if* $\beta \in U$.

(vii) $\log \mathrm{Tree}(\alpha + \beta) = \max\{\log \mathrm{Tree}(\alpha), \log \mathrm{Tree}(\beta)\}$.

(viii) $\log \mathrm{Tree}(\alpha\beta) = \log \mathrm{Tree}(\alpha) + \log \mathrm{Tree}(\beta)$.

(ix) $\log^2 \mathrm{Tree}(\alpha + \beta) = \max\{\log^2 \mathrm{Tree}(\alpha), \log^2 \mathrm{Tree}(\beta)\}$.

(x) $\log^2 \mathrm{Tree}(\alpha\beta) \leq \max\{\log^2 \mathrm{Tree}(\alpha), \log^2 \mathrm{Tree}(\beta)\}$ *and they are equal if* $\alpha, \beta \neq 0$.

(xi) If $\alpha \neq 0$, $\mathrm{Tree}(\alpha^\diamond) = \exp^2 \mathrm{Tree}(\alpha)$.

(xii) If $\alpha \neq 0$, $\mathrm{Tree}(\alpha^\diamond) > \log^2 \mathrm{Tree}(\alpha^\diamond) = \mathrm{Tree}(\alpha)$.

(xiii) If $\alpha \in U \natural X$, *then* $\mathrm{width}(\mathrm{Tree}(\alpha)) = 1$.

Proof. Property (xi) holds because, when restricted to $\mathrm{Rat}(U)$, Tree is a morphism of rational U-semirings. Properties (i)-(x) and (xii) are proved in [Sán08, Lemma 5.40], except for the inequality $\mathrm{Tree}(\alpha^\diamond) > \mathrm{Tree}(\alpha)$, which follows from property (xi) since $\mathrm{height}(\mathrm{Tree}(\alpha^\diamond)) > \mathrm{height}(\mathrm{Tree}(\alpha))$ and the ordering in $\mathcal{T}$ refines the ordering by height. Property $(xiii)$ is observed in [Sán08, page 112]: if $\alpha \in U$, then $\mathrm{Tree}(\alpha) = 1_{\mathcal{T}}$ and consequently $\mathrm{width}(\mathrm{Tree}(\alpha)) = 1$; if $\alpha \in X$, then since $\diamond$ carries bijectively $\mathrm{Rat}(U)$ to X, we have $\alpha = \beta^\diamond$ for some $\beta \in \mathrm{Rat}(U)$, and therefore by the property (xi), we have $\mathrm{width}(\mathrm{Tree}(\alpha)) = \mathrm{width}(\exp^2(\mathrm{Tree}(\beta))) = 1$; finally, if $\alpha \in U\natural X \setminus (U \cup X)$, then $\alpha = x_1 \ldots x_n$ with $x_i \in X$ for some $n \geq 2$, and since Tree is a morphism of semirings and the width of a product is the product of the widths (there are $\mathrm{width}(X)\mathrm{width}(Y)$ elements in $\mathrm{fam}(X \cdot Y)$), we obtain as claimed $\mathrm{width}(\mathrm{Tree}(\alpha)) = \prod_i \mathrm{width}(\mathrm{Tree}(x_i)) = 1$. $\square$

A crucial step for the inductive method used in [DHS04] is that they can construct recursively, for every element α in $\mathrm{Rat}(U)$, a finitely generated subgroup $\mathrm{source}(\alpha)$ of U which is the (unique) smallest with the property that $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha))U$. The construction and main properties of this object can be found in [Sán08, Section 5.5], and here we list some of them (see [Sán08, Lemmas 5.42, 5.44 and 5.47, Definition 5.45 & Remark 5.46]).

Theorem 4.3.8. *Let U be a multiplicative group. The following hold.*

(i) *There is a subset P of $\mathrm{Rat}(U)$, whose elements are called* primitive, *which is closed under U-conjugation and satisfies $PU = UP = \mathrm{Rat}(U)$.*

(ii) *For $p \in P$, there exists a finitely generated subgroup* $\mathrm{source}_U(p)$ *of U such that $p \in \mathrm{Rat}(\mathrm{source}_U(p))$. Morever, if U is a subgroup of W, then p is primitive over* $\mathrm{Rat}(W)$ *and* $\mathrm{source}_U(p) = \mathrm{source}_W(p)$. *For this reason, we can just write* $\mathrm{source}(p)$.

(iii) *For $\alpha \in \mathrm{Rat}(U)$, $\alpha = pu$ for some $p \in P$, $u \in U$, we can set* $\mathrm{source}(\alpha) = \mathrm{source}(p)$ *and this does not depend on the choice of p, u. In particular,* $\mathrm{source}(\alpha)$ *is a finitely generated subgroup of U and $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha)) \cdot U$.*

(iv) *If $\alpha \in \mathrm{Rat}(U)$ and V is a subgroup of U such that $\alpha \in \mathrm{Rat}(V) \cdot U$, then* $\mathrm{source}(\alpha) \leq V$.

4.3.3 Division $E * G$-closures

The following example is a modification of [Sán08, Example 1.43(d)]. Let us fix throughout this section a division ring E, a group G, and a crossed product $E * G$, where we assume that $1_{E*G} = u_e$.

Let $(\mathcal{A}, \phi)$ be an $E * G$-ring, i.e., $\phi : E * G \to \mathcal{A}$ is a ring homomorphism. For each subgroup H of G, denote the division closure of $\phi(E * H)$ in $\mathcal{A}$ by $\mathcal{D}_{H,\mathcal{A}}$. Since $E^\times H$ is a subgroup of the group of units of $E * H$ (see Corollary 3.4.4(iv)), and $E^\times H \hookrightarrow E * H \xrightarrow{\phi} \mathcal{D}_{H,\mathcal{A}}$ is a group homomorphism, $\mathcal{D}_{H,\mathcal{A}}$ is an $E^\times H$-semiring. Now set, for $a \in \mathcal{D}_{H,\mathcal{A}}$,

$$a^\diamond = \begin{cases} a^{-1} & \text{if } a \text{ is invertible in } \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

The $\diamond$-map is well-defined because $\mathcal{D}_{H,\mathcal{A}}$ is division closed. Now, for every $u, v \in E^\times H$ we have, if $a \in \mathcal{D}_{H,\mathcal{A}}$ is invertible,

$$\phi(v^{-1})a^\diamond\phi(u^{-1}) = \phi(v)^{-1}a^{-1}\phi(u)^{-1} = (\phi(u)a\phi(v))^{-1} = (\phi(u)a\phi(v))^\diamond,$$

and if a is not invertible, then neither is $\phi(u)a\phi(v)$, and hence

$$\phi(v^{-1})a^\diamond\phi(u^{-1}) = 0 = (\phi(u)a\phi(v))^\diamond.$$

Therefore, we have defined an $E^\times H$-rational structure on $\mathcal{D}_{H,\mathcal{A}}$, and by Lemma 4.3.6 applied to $\mathcal{D}_{H,\mathcal{A}}$ we obtain a unique morphism of rational $E^\times H$-semirings

$$\Phi_{H,\mathcal{A}} : \mathrm{Rat}(E^\times H) \to \mathcal{D}_{H,\mathcal{A}}$$

that can be further extended to a morphism of $E^\times H$-semirings

$$\Phi'_{H,\mathcal{A}} : \mathrm{Rat}(E^\times H) \cup \{0\} \to \mathcal{D}_{H,\mathcal{A}}.$$

The following proposition is a rewriting of [Sán08, Examples 5.37 and 5.38].

Proposition 4.3.9. *For every subgroup $H \leq G$, the morphism of rational U-semirings*

$$\Phi_{H,\mathcal{A}} : \mathrm{Rat}(E^\times H) \to \mathcal{D}_{H,\mathcal{A}}$$

is surjective, and the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Rat}(E^\times H) \arrow[r, "\Phi_{H,\mathcal{A}}"] \arrow[d, hook] & \mathcal{D}_{H,\mathcal{A}} \arrow[d, hook] \\
\mathrm{Rat}(E^\times G) \arrow[r, "\Phi_{G,\mathcal{A}}"'] & \mathcal{D}_{G,\mathcal{A}}
\end{tikzcd}$$

Proof. Set $U = E^\times H$, $1_{E*H} = 1$ and let Q_n denote the n-th step in the construction of $\mathcal{D}_{H,\mathcal{A}}$ as in Proposition 3.3.2 , with $Q_0 = \phi(E * H)$. For the first part, set $\Phi = \Phi_{H,\mathcal{A}}$ to ease the notation. We claim that $\Phi(\mathbb{N}[U]\backslash\{0\}) = Q_0$ and that $\Phi(\mathbb{N}[U\natural X_n]\backslash\{0\}) = Q_n$ for every $n \geq 1$.

For $n = 0$, observe from the construction of Φ in Lemma 4.3.6 that $\Phi(\sum n_u u) = \sum n_u \phi(u)$ for every element in $\mathbb{N}[U]\backslash\{0\}$. This implies that $\Phi(\mathbb{N}[U]\backslash\{0\}) \subseteq Q_0$. Conversely, since every non-zero element in $E * H$ is a finite sum of elements in $E^\times H$ then $Q_0\backslash\{0\} \subseteq \Phi(\mathbb{N}[U]\backslash\{0\})$. Finally, since, $-1 \in U$, then $1 + (-1) \in \mathbb{N}[U]\backslash\{0\}$ and $\Phi(1 + (-1)) = \phi(1) + \phi(-1) = 0$, since ϕ comes from a ring homomorphism. Therefore, $Q_0 = \Phi(\mathbb{N}[U]\backslash\{0\})$.

Set $n = 1$. Since Φ preserves the rational structure and every element in X_1 is of the form $x = y^\diamond$ for some $y \in \mathbb{N}[U]\backslash\{0\}$, then $\Phi(x) = \Phi(y)^\diamond$ with $\Phi(y) \in \Phi(\mathbb{N}[U]\backslash\{0\}) = Q_0$. Since $\Phi(y)^\diamond$ is either zero or the inverse of an element of Q_0, we have by definition $\Phi(X_1) \subseteq Q_1$. Now the elements in $U\natural X_1$, resp. $\mathbb{N}[U\natural X_1]\backslash\{0\}$, are obtained from the previous ones by means of products and sums, and therefore $\Phi(\mathbb{N}[U\natural X_1]\backslash\{0\}) \subseteq Q_1$. Conversely, let $a \in Q_1$ be such that $a = b^{-1}$ for some $b \in Q_0 = \Phi(\mathbb{N}[U]\backslash\{0\})$. In particular, b is non-zero and $b = \Phi(y)$ for some $y \in \mathbb{N}[U]\backslash\{0\}$. As a consequence,

$$a = b^{-1} = b^\diamond = \Phi(y)^\diamond = \Phi(y^\diamond)$$

and $y^\diamond \in X_1$. In this way, every generator of Q_1 lies in the image of $X_1 \cup \mathbb{N}[U]\backslash\{0\} \subseteq \mathbb{N}[U\natural X_1]\backslash\{0\}$. Since $-1 \in U$, $\mathbb{N}[U\natural X_1]\backslash\{0\}$ is a U-semiring, Φ is a morphism of rational U-semirings and any other non-zero element in Q_1 is constructed by means of sums, substractions and products of the generators, we see that $Q_1\backslash\{0\} \subseteq \Phi(\mathbb{N}[U\natural X_1]\backslash\{0\})$. Since we already have a preimage for 0, we obtain equality.

Assume $n \geq 2$ and that we have proved $\Phi(\mathbb{N}[U\natural X_n]\backslash\{0\}) = Q_n$. As before, every element in X_{n+1} has the form $x = y^\diamond$ for some $y \in \mathbb{N}[U\natural X_n]\backslash\{0\}$, by the induction hypothesis we get $\Phi(X_{n+1}) \subseteq Q_{n+1}$, and consequently $\Phi(\mathbb{N}[U\natural X_{n+1}]) \subseteq Q_{n+1}$. Conversely, the generators of Q_{n+1} lie now in the image of $X_{n+1} \cup \mathbb{N}[U \natural X_n] \setminus \{0\} \subseteq \Phi(\mathbb{N}[U \natural X_{n+1}] \setminus \{0\})$, and hence using the same argument we can construct a preimage in $\mathbb{N}[U \natural X_{n+1}] \setminus \{0\}$ for every element in Q_{n+1}, from where equality follows.

Therefore, we see that for every element in $\mathcal{D}_{H,\mathcal{A}} = \bigcup_n Q_n$, there exists a preimage in $\mathrm{Rat}(U)$, and hence Φ is surjective.

For the second part of the statement, the left inclusion of the diagram is actually given by the injective morphism of rational $E^\times H$-semirings $\Psi_{E^\times H, E^\times G}$. Now $\mathcal{D}_{G,\mathcal{A}}$ is a rational $E^\times H$-semiring and the right inclusion is a morphism of $E^\times H$-semirings. Moreover, from the definition of the $\diamond$-map we can see that if $a \in \mathcal{D}_{H,\mathcal{A}}$, then $a^\diamond$ is the same whether we consider it over $\mathcal{D}_{H,\mathcal{A}}$ or over $\mathcal{D}_{G,\mathcal{A}}$ (because $\mathcal{D}_{H,\mathcal{A}}$ is division closed). Hence, both paths of the diagram from $\mathrm{Rat}(E^\times H) \to \mathcal{D}_{G,\mathcal{A}}$ define morphisms of rational $E^\times H$-semirings, and hence the uniqueness in Lemma 4.3.6 implies that they are equal. $\square$

We can now define the notion of H-complexity for elements in $\mathcal{D}_{H,\mathcal{A}}$.

Definition 4.3.10. Let H be a subgroup of G and take $a \in \mathcal{D}_{H,\mathcal{A}}$. We set

$$\mathrm{Tree}_H(a) = \min\{\mathrm{Tree}(\alpha) : \alpha \in \mathrm{Rat}(E^\times H) \cup \{0\}, \Phi'_{H,\mathcal{A}}(\alpha) = a\}.$$

and analogously,

$$\mathrm{Tree}_G(a) = \min\{\mathrm{Tree}(\alpha) : \alpha \in \mathrm{Rat}(E^\times G) \cup \{0\}, \Phi'_{G,\mathcal{A}}(\alpha) = a\}.$$

We say that $\alpha \in \mathrm{Rat}(E^\times H) \cup \{0\}$ *realizes the H-complexity of* a if it satisfies $\Phi_{H,\mathcal{A}}(\alpha) = a$ and $\mathrm{Tree}(\alpha) = \mathrm{Tree}_H(a)$. Similarly for the G-complexity.

This notion is well-defined because $\mathcal{T}$ is well-ordered by Lemma 4.3.3. Notice from the definition that, if $a, b \in \mathcal{D}_{H,\mathcal{A}}$ and $u \in E^\times H$, then we always have

$$\mathrm{Tree}_H(ab) \leq \mathrm{Tree}_H(a)\mathrm{Tree}_H(b)$$

$$\mathrm{Tree}_H(a+b) \leq \mathrm{Tree}_H(a) + \mathrm{Tree}_H(b)$$

$$\mathrm{Tree}_H(a) = \mathrm{Tree}_H(a\phi(u)) = \mathrm{Tree}_H(\phi(u)a)$$

$$\mathrm{Tree}_H(a) = \mathrm{Tree}_H(-a)$$

Indeed, if α and β realize the H-complexity of a and b, respectively, then $\Phi'_{H,\mathcal{A}}(\alpha\beta) = ab$ and $\Phi'_{H,\mathcal{A}}(\alpha + \beta) = a + b$, because $\Phi'_{H,\mathcal{A}}$ is a morphism of semirings. By definition, and using Lemma 4.3.7(iii) and (v), we deduce

$$\mathrm{Tree}_H(ab) \leq \mathrm{Tree}(\alpha\beta) \leq \mathrm{Tree}(\alpha)\mathrm{Tree}(\beta) = \mathrm{Tree}_H(a)\mathrm{Tree}_H(b)$$

and

$$\mathrm{Tree}_H(a+b) \leq \mathrm{Tree}(\alpha+\beta) \leq \mathrm{Tree}(\alpha) + \mathrm{Tree}(\beta) = \mathrm{Tree}_H(a) + \mathrm{Tree}_H(b).$$

Similarly, if α_1 is such that $\Phi'_{H,\mathcal{A}}(\alpha_1) = a$, then $\alpha_1 u$ has the same complexity by Lemma 4.3.7(v) and (ii), and $\Phi'_{H,\mathcal{A}}(\alpha_1 u) = a\phi(u)$ because $\Phi'_{H,\mathcal{A}}$ is a morphism of

$E^\times H$-semirings. Conversely, if α_2 satisfies $\Phi'_{H,\mathcal{A}}(\alpha_2) = a\phi(u)$, then $\alpha_2 u^{-1}$ has the same complexity and $\Phi'_{H,\mathcal{A}}(\alpha_2 u^{-1}) = a\phi(u)\phi(u^{-1}) = a$. We deduce from here that $\mathrm{Tree}_H(a) = \mathrm{Tree}_H(a\phi(u))$, and a similar argument shows the other identity. The last equality is a particular case of the previous one taking into account that $-1 \in E^\times H$ and $\phi(-1) = -1$ because ϕ is the restriction of the original ring homomorphism $\phi : E{*}G \to \mathcal{A}$.

In addition, writing things carefully, we have shown through the previous subsection and Proposition 4.3.9 that we have a commutative diagram of morphisms of rational $E^\times H$-semirings

$$\begin{tikzcd}
 & \mathrm{Rat}(E^\times H) \arrow[r, "\Phi_{H,\mathcal{A}}"] \arrow[dl, "\mathrm{Tree}_{E^\times H}"'] \arrow[dd, "\Psi_{E^\times H, E^\times G}"] & \mathcal{D}_{H,\mathcal{A}} \arrow[dd, hook] \\
\mathcal{T} & & \\
 & \mathrm{Rat}(E^\times G) \arrow[r, "\Phi_{G,\mathcal{A}}"'] \arrow[ul, "\mathrm{Tree}_{E^\times G}"] & \mathcal{D}_{G,\mathcal{A}}
\end{tikzcd}$$

In particular, if $a \in \mathcal{D}_{H,\mathcal{A}}$ and $\alpha \in \mathrm{Rat}(E^\times H)$ is such that $\Phi_{H,\mathcal{A}}(\alpha) = a$, then $\alpha' = \Psi_{E^\times H, E^\times G}(\alpha)$ is an element in $\mathrm{Rat}(E^\times G)$ satisfying

$$\Phi_{G,\mathcal{A}}(\alpha') = \Phi_{G,\mathcal{A}}(\Psi_{E^\times H,E^\times G}(\alpha)) = \Phi_{H,\mathcal{A}}(\alpha) = a$$

and

$$\mathrm{Tree}_{E^\times G}(\alpha') = \mathrm{Tree}_{E^\times G}(\Psi_{E^\times H,E^\times G}(\alpha)) = \mathrm{Tree}_{E^\times H}(\alpha).$$

In other words, we always have $\mathrm{Tree}_G(a) \le \mathrm{Tree}_H(a)$. In view of the commutativity of the diagram, we shall sometimes identify $\mathrm{Rat}(E^\times H) \subseteq \mathrm{Rat}(E^\times G)$, write $\mathrm{Tree}_{E^\times H}$ and $\mathrm{Tree}_{E^\times G}$ simply as Tree, and $\Phi_{H,\mathcal{A}}$ and $\Phi_{G,\mathcal{A}}$ simply as Φ. In this sense, talking about the H or G-complexity of an element should be sufficient to identify which morphisms we are using.

The following is an important remark.

Lemma 4.3.11. *Let $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$ be $E * G$-rings, and assume that $\mathcal{D}_{G,\mathcal{A}_1}$ and $\mathcal{D}_{G,\mathcal{A}_2}$ are $E * G$-isomorphic, i.e., there exists an isomorphism $\varphi : \mathcal{D}_{G,\mathcal{A}_1} \to \mathcal{D}_{G,\mathcal{A}_2}$ such that the following commutes*

$$\begin{tikzcd}
 & E * G \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
\mathcal{D}_{G,\mathcal{A}_1} \arrow[rr, "\varphi"'] & & \mathcal{D}_{G,\mathcal{A}_2}
\end{tikzcd}$$

Then we have $\mathrm{Tree}_G(a) = \mathrm{Tree}_G(\varphi(a))$ *for every $a \in \mathcal{D}_{G,\mathcal{A}_1}$ and the elements realizing their complexity in* $\mathrm{Rat}(E^\times G) \cup \{0\}$ *are the same.*

Proof. Since φ is a homomorphism of $E * G$-semirings, we have, for every $u, v \in E^\times G$, that

$$\varphi(\phi_1(u)a\phi_1(v)) = \varphi(\phi_1(u))\varphi(a)\varphi(\phi_1(v)) = \phi_2(u)\varphi(a)\phi_2(v),$$

i.e., it is a morphism of $E^\times G$-semirings. Moreover, since it is an isomorphism, we have that $a \in \mathcal{D}_{G,\mathcal{A}_1}$ is invertible if and only if $\varphi(a)$ is invertible. As a consequence, $\varphi(a^\diamond) = \varphi(a)^\diamond$ and hence φ is a morphism of rational $E^\times G$-semirings. Consider the diagram

$$\begin{tikzcd} & \mathrm{Rat}(E^\times G) \arrow[dl, "\Phi_{G,\mathcal{A}_1}"'] \arrow[dr, "\Phi_{G,\mathcal{A}_2}"] & \\ \mathcal{D}_{G,\mathcal{A}_1} \arrow[rr, "\varphi"'] & & \mathcal{D}_{G,\mathcal{A}_2}, \end{tikzcd}$$

where $\Phi_{G,\mathcal{A}_1}$ and $\Phi_{G,\mathcal{A}_2}$ are the universal morphisms from Lemma 4.3.6. Since $\varphi \circ \Phi_{G,\mathcal{A}_1}$ is a morphism of rational $E^\times G$-semirings, we must have by uniqueness that $\varphi \circ \Phi_{G,\mathcal{A}_1} = \Phi_{G,\mathcal{A}_2}$.

Consider any $a \in \mathcal{D}_{G,\mathcal{A}_1}$. Since φ is an isomorphism, $a = 0$ if and only if $\varphi(a) = 0$, and in such case $\mathrm{Tree}_G(a) = \mathrm{Tree}_G(\varphi(a)) = 0_{\mathcal{T}}$. If $a \neq 0$ and $\alpha \neq 0$ is such that $\Phi_{G,\mathcal{A}_1}(\alpha) = a$, then from the commutativity of the diagram, $\Phi_{G,\mathcal{A}_2}(\alpha) = \varphi(\Phi_{G,\mathcal{A}_1}(\alpha)) = \varphi(a)$, and hence $\mathrm{Tree}_G(\varphi(a)) \leq \mathrm{Tree}_G(a)$. Similarly, if $\beta \neq 0$ is such that $\Phi_{G,\mathcal{A}_2}(\beta) = \varphi(a)$, then we have that $\Phi_{G,\mathcal{A}_1}(\beta) = \varphi^{-1}(\Phi_{G,\mathcal{A}_2}(\beta)) = a$, and this gives us the other inequality. Thus, for every a, $\mathrm{Tree}_G(a) = \mathrm{Tree}_G(\varphi(a))$. The commutativity of the diagram then also shows the last assertion in the theorem, since $\Phi_{G,\mathcal{A}_1}(\alpha) = a$ if and only if $\Phi_{G,\mathcal{A}_2}(\alpha) = \varphi(a)$. $\square$

Let us now explain more precisely the situation we are interested in and how the notion of complexity will be used for the proof of the Atiyah conjecture for locally indicable groups. For this, assume that in the given crossed product $E * G$, G is locally indicable, and let H, N be, respectively, a finitely generated subgroup of G and a normal subgroup $N \triangleleft H$ such that H/N is infinite cyclic. Consider $\pi_H : E^\times H \to E^\times H/E^\times \cong H$ and let $x \in E^\times H$ be such that $H/N = \langle N\pi_H(x)\rangle$.

On the one hand, we have already seen (see, for instance, Lemma 3.4.20) that left conjugation by x induces a ring automorphism τ of $E * N$ because x normalizes $E * N$, i.e., $x(E * N)x^{-1} = E * N$, and moreover we have $E * H \cong E * N[t^{\pm 1};\tau]$. On the other hand, consider the identification $\mathrm{Rat}(E^\times N) \subseteq \mathrm{Rat}(E^\times H)$. Since $x(E^\times N)x^{-1} = E^\times N$, left conjugation by x can also be extended to a semiring automorphism of $\mathrm{Rat}(E^\times N)$ (cf. [Sán08, Page 108]). Indeed, $\mathrm{Rat}(E^\times N)$ is an $E^\times N$-semiring, and hence for every $\alpha \in \mathrm{Rat}(E^\times N), u, v \in E^\times N$,

$$\left[(xux^{-1})\alpha(xvx^{-1})\right]^\diamond = (xvx^{-1})^{-1}\alpha^\diamond(xux^{-1})^{-1}.$$

Thus, the composition $E^\times N \xrightarrow{\tau} E^\times N \to \mathrm{Rat}(E^\times N)$ defines a new structure of rational $E^\times N$-semiring on $\mathrm{Rat}(E^\times N)$. Using its universal property, there exists a unique morphism of rational $E^\times N$-semirings

$$\bar{\tau} : \mathrm{Rat}(E^\times N) \to \mathrm{Rat}(E^\times N)$$

that acts as τ on elements of $E^\times N$. We can similarly extend $\upsilon = \tau^{-1}$ (i.e., left conjugation by x^{-1}), and since $\tau\tau^{-1} = \tau^{-1}\tau = \mathrm{id}_{E^\times N}$, the uniqueness in the universal property of $\mathrm{Rat}(E^\times N)$ tells us that $\bar{\tau}\bar{\upsilon} = \bar{\upsilon}\bar{\tau} = \mathrm{id}_{\mathrm{Rat}(E^\times N)}$. Although the domain and codomain of $\bar{\tau}$ differ in the $E^\times N$-biset structure, the underlying rational semiring is the same, and hence forgetting about these structures $\bar{\tau}$ is a semiring automorphism of $\mathrm{Rat}(E^\times N)$ respecting the $\diamond$-map. From its construction in Lemma 4.3.6 we see that, in $\mathrm{Rat}(E^\times H)$, $\bar{\tau}(r)x = xr$ for every $r \in \mathrm{Rat}(E^\times N)$.

Let us denote by $\mathrm{Rat}(E^\times N)\langle x\rangle$ the subset of $\mathrm{Rat}(E^\times H)$ consisting of the elements αx^n for some $\alpha \in \mathrm{Rat}(E^\times N)$ and $n \in \mathbb{Z}$. This is actually a multiplicative submonoid of $\mathrm{Rat}(E^\times H)$, since $1_{\mathrm{Rat}(E^\times H)} = 1_{\mathrm{Rat}(E^\times N)}$ and for every $\alpha, \beta \in \mathrm{Rat}(E^\times N)$, $n, m \in \mathbb{Z}$, we have

$$(\alpha x^n) \cdot (\beta x^m) = \alpha\bar{\tau}^n(\beta)x^{n+m} \in \mathrm{Rat}(E^\times N)\langle x\rangle.$$

Note in particular that the following hold.

- If $\alpha \in \mathrm{Rat}(E^\times N)$, then $x^n\alpha = \bar{\tau}^n(\alpha)x^n \in \mathrm{Rat}(E^\times N)\langle x\rangle$.
- If $\alpha, \beta \in \mathrm{Rat}(E^\times N)\langle x\rangle$, then $\alpha\beta \in \mathrm{Rat}(E^\times N)\langle x\rangle$.
- If $\alpha, \beta \in \mathrm{Rat}(E^\times N)x^n$, then $\alpha + \beta \in \mathrm{Rat}(E^\times N)x^n$.

A key auxiliary result

After this, we are going to prove the key result regarding the induction on the complexity of elements. The starting point is the one described above, i.e, N is a normal subgroup of a group H such that H/N is infinite cyclic, $E * H$ is a crossed product with corresponding subcrossed product $E * N$, and τ denotes the automorphism of $E * N$ induced by left conjugation by an element $x \in E^\times H$ whose image under the composition $E^\times H \to E^\times H/E^\times \cong H \to H/N$ generates H/N. The general context we need in order to apply the result is the following.

(i) A (von Neumann) regular $E * N$-ring $(\mathcal{A}, \phi)$.

(ii) An automorphism $\tilde{\tau}$ of $\mathcal{A}$ such that the following diagram commutes.

$$\begin{tikzcd} E * N \arrow[r, "\tau"] \arrow[d, "\phi"'] & E * N \arrow[d, "\phi"] \\ \mathcal{A} \arrow[r, "\tilde{\tau}"'] & \mathcal{A} \end{tikzcd}$$

(iii) A ring $\mathcal{P}$ with an embedding $\mathcal{A}((t; \tilde{\tau})) \xrightarrow{f} \mathcal{P}$.

Under these hypothesis:

Remark 4.3.12.

1. We can extend ϕ to a map $\tilde{\phi} : E * H \to \mathcal{A}((t;\tilde{\tau}))$ by sending $x \mapsto t$. Indeed, since $E * H = \bigoplus_{i\in\mathbb{Z}}(E * N)x^i$, every element in $E * H$ is uniquely written as $p = \sum a_i x^i$ for $a_i \in E * N$, from where $\tilde{\phi}$ is well-defined and sends $p \mapsto \sum \phi(a_i)t^i$. From the definition $\tilde{\phi}$ is linear and note that condition (ii) ensures that it is a ring homomorphism, because for all $a, b \in E * N$, $(ax^i)(bx^j) = a\tau^i(b)x^{i+j}$ and consequently

$$\begin{aligned}\tilde{\phi}((ax^i)(bx^j)) &= \phi(a)\phi(\tau^i(b))t^{i+j} = \phi(a)\tilde{\tau}^i(\phi(b))t^{i+j} \\ &= [\phi(a)t^i][\phi(b)t^j] = \tilde{\phi}(ax^i)\tilde{\phi}(bx^j).\end{aligned}$$

2. Because of the previous step, we have

$$\begin{tikzcd} E * N \arrow[r, hook] \arrow[d, "\phi"'] & E * H \arrow[d, "\tilde{\phi}"] & \\ \mathcal{A} \arrow[r, hook] & \mathcal{A}((t;\tilde{\tau})) \arrow[r, "f"] & \mathcal{P}, \end{tikzcd}$$

   and hence we can consider $\mathcal{D}_{N,\mathcal{P}}$ and $\mathcal{D}_{H,\mathcal{P}}$, the division closures of $f\phi(E * N)$ and $f\tilde{\phi}(E*H)$ inside $\mathcal{P}$. Let $\mathcal{D}_{N,\mathcal{A}}$ denote the division closure of $\phi(E*N)$ in $\mathcal{A}$. Since $\mathcal{A}$ is regular by condition (i) and f is injective by condition (iii), we have that $f(\mathcal{A})$ is regular and by Lemma 3.3.3(2) and (3), we deduce that f defines an isomorphism $\mathcal{D}_{N,\mathcal{A}} \cong \mathcal{D}_{N,f(\mathcal{A})} = \mathcal{D}_{N,\mathcal{P}}$ as $E * N$-rings.

3. The restriction of $\tilde{\tau}$ to $\mathcal{D}_{N,\mathcal{A}}$ is an automorphism of $\mathcal{D}_{N,\mathcal{A}}$. Indeed, since $\tilde{\tau}$ is an automorphism of $\mathcal{A}$, the same Lemma 3.3.3(3) tells us that $\tilde{\tau}$ defines an isomorphism between $\mathcal{D}_{N,\mathcal{A}}$ and the division closure of $\tilde{\tau}\phi(E*N)$ in $\mathcal{A}$. But from (ii), $\tilde{\tau}\phi(E*N) = \phi\tau(E*N) = \phi(E*N)$ because τ is an automorphism of $E*N$, and hence the latter division closure equals $\mathcal{D}_{N,\mathcal{A}}$, what proves the claim.

   This means that we can talk about $\mathcal{D}_{N,\mathcal{A}}((t;\tilde{\tau}))$, a subring of $\mathcal{A}((t;\tilde{\tau}))$.

4. As we did before, given the identification $\mathrm{Rat}(E^\times N) \subseteq \mathrm{Rat}(E^\times H)$, τ extends to a semiring automorphism $\bar{\tau}$ of $\mathrm{Rat}(E^\times N)$ and we can consider the multiplicative submonoid $\mathrm{Rat}(E^\times N)\langle x\rangle$ of $\mathrm{Rat}(E^\times H)$.

   We denote by Φ (instead of $\Phi_{H,\mathcal{P}}$) the unique morphism

$$\Phi : \mathrm{Rat}(E^\times H) \to \mathcal{D}_{H,\mathcal{P}}$$

   of rational $E^\times H$-semirings given by Lemma 4.3.6, and by Φ' its extension to $\mathrm{Rat}(E^\times H) \cup \{0\}$. Note that given the $E^\times H$-structure of $\mathcal{D}_{H,\mathcal{P}}$, for $u \in E^\times H$ we have $\Phi(u) = \tilde{\phi}(u)$.

□

We are in position to state and prove the key result, which mimics the strategy of [Sán08, Theorem 5.49].

Proposition 4.3.13. *Assume conditions (i),(ii) and (iii) are satisfied and adopt the previous notation. Take $a \in \mathcal{D}_{H,\mathcal{P}}$ and assume that every non-zero $c \in \mathcal{D}_{H,\mathcal{P}}$ such that* $\mathrm{Tree}_H(c) < \mathrm{Tree}_H(a)$ *is invertible in $\mathcal{D}_{H,\mathcal{P}}$. Then for every $b \in \mathcal{D}_{H,\mathcal{P}}$ such that* $\mathrm{Tree}_H(b) \leq \mathrm{Tree}_H(a)$ *the following hold.*

1. *$b = f(\bar{b})$ for some $\bar{b} \in \mathcal{D}_{N,\mathcal{A}}((t;\tilde{\tau}))$ and*
2. *if b is non-zero and $\bar{b} = \sum b_k$ with $b_k \in \mathcal{D}_{N,\mathcal{A}}t^k$, then*

$$\mathrm{Tree}_H(f(b_k)) \leq \mathrm{Tree}_H(b)$$

for all k, and equality holds for some n if and only if $\bar{b} = b_n \in \mathcal{D}_{N,\mathcal{A}}t^n$ and

$$\left\{\beta \in \mathrm{Rat}(E^\times H) : \Phi(\beta) = b \text{ and } \mathrm{Tree}(\beta) = \mathrm{Tree}_H(b)\right\} \subseteq \mathrm{Rat}(E^\times N)x^n.$$

Proof. For $b = 0$ there is nothing to prove, so let $b \neq 0$. If $\mathrm{Tree}_H(b) = 1_{\mathcal{T}}$ and β realizes the H-complexity of b, then $\beta \in E^\times H$ by Lemma 4.3.7(ii), and hence we can write $\beta = ax^n$ for some $a \in E^\times N$ and $n \in \mathbb{Z}$. Thus, $\beta \in \mathrm{Rat}(E^\times N)x^n$ and $b = \Phi(\beta) = f\tilde{\phi}(ax^n) = f(\phi(a)t^n) = f(\bar{b})$ where $\bar{b} = b_n = \phi(a)t^n \in \mathcal{D}_{N,\mathcal{A}}t^n$, so the result holds.

Suppose now that $\mathrm{Tree}_H(b) > 1_{\mathcal{T}}$ and that the result holds for every element $c \in \mathcal{D}_{H,\mathcal{P}}$ with $\mathrm{Tree}_H(c) < \mathrm{Tree}_H(b)$. Fix an arbitrary element $\beta \in \mathrm{Rat}(E^\times H)$ realizing the H-complexity of b (β is non-zero because b is non-zero). We are going to divide $\mathrm{Rat}(E^\times H)$ in four disjoint subsets

$$U = E^\times H \qquad X \qquad U\natural X\backslash(X \cup U) \qquad \mathbb{N}[U\natural X]\backslash(U\natural X \cup \{0\}).$$

As far as we are assuming $\mathrm{Tree}_H(b) > 1_{\mathcal{T}}$, we know that $\beta \notin U$ again by Lemma 4.3.7(ii), so we have three possibilities left:

Case 1. If $\beta \in \mathbb{N}[U\natural X]\backslash(U\natural X \cup \{0\})$, then there exist $\gamma, \delta \in \mathbb{N}[U\natural X]\backslash\{0\}$ such that $\beta = \gamma + \delta$. By Lemma 4.3.7(iv),

$$\mathrm{Tree}(\gamma), \mathrm{Tree}(\delta) < \mathrm{Tree}(\beta).$$

Setting $c = \Phi(\gamma), d = \Phi(\delta)$, we obtain a decomposition $b = c + d$. We claim that γ realizes the H-complexity of c, i.e., $\mathrm{Tree}_H(c) = \mathrm{Tree}(\gamma)$. Otherwise, there would exist $\gamma' \in \mathrm{Rat}(E^\times H) \cup \{0\}$ with $\Phi'(\gamma') = c$ satisfying $\mathrm{Tree}(\gamma') < \mathrm{Tree}(\gamma)$, from where using Lemma 4.3.7(iii) and Lemma 4.3.4(i)

$$\begin{aligned}\mathrm{Tree}(\gamma' + \delta) &= \mathrm{Tree}(\gamma') + \mathrm{Tree}(\delta) < \mathrm{Tree}(\gamma) + \mathrm{Tree}(\delta)\\ &= \mathrm{Tree}(\gamma + \delta) = \mathrm{Tree}(\beta).\end{aligned}$$

Since $\Phi'(\gamma' + \delta) = b$, this contradicts the minimality of β. Similarly, δ realizes the H-complexity of d, and therefore we have found a decomposition $b = c + d$ with $\mathrm{Tree}_H(b) >$

$\mathrm{Tree}_H(c), \mathrm{Tree}_H(d)$. In particular, since the elements realizing the H-complexities of c and d (i.e., γ and δ) are non-zero, c and d must be non-zero (otherwise, 0 would be the unique element realizing their H-complexity by Lemma 4.3.7(i)).

Now, by the induction hypothesis, we can write $c = f(\bar{c})$, $d = f(\bar{d})$ where $\bar{c} = \sum c_n$, $\bar{d} = \sum d_n$ with $c_n, d_n \in \mathcal{D}_{N,\mathcal{A}} t^n$, $\mathrm{Tree}_H(f(c_n)) \leq \mathrm{Tree}_H(c)$ and $\mathrm{Tree}_H(f(d_n)) \leq \mathrm{Tree}_H(d)$. Hence, we have an expression $\bar{b} = \bar{c} + \bar{d} = \sum b_n$ with $b_n = c_n + d_n$ for every n, and $f(\bar{b}) = f(\bar{c}) + f(\bar{d}) = c + d = b$. Let $\beta_n, \gamma_n, \delta_n$ be elements in $\mathrm{Rat}(E^\times H) \cup \{0\}$ realizing the H-complexities of $f(b_n), f(c_n), f(d_n)$, respectively, for all n. From the previous expression of b_n (and hence of $f(b_n)$) and the behavior of Tree_H with respect to sums we obtain

$$\begin{aligned}
\mathrm{Tree}(\beta_n) &= \mathrm{Tree}_H(f(b_n)) \leq \mathrm{Tree}_H(f(c_n)) + \mathrm{Tree}_H(f(d_n)) \\
&= \mathrm{Tree}(\gamma_n) + \mathrm{Tree}(\delta_n) \\
&\leq \mathrm{Tree}_H(c) + \mathrm{Tree}_H(d) \\
&= \mathrm{Tree}(\gamma) + \mathrm{Tree}(\delta) \\
&= \mathrm{Tree}(\beta) = \mathrm{Tree}_H(b),
\end{aligned}$$

where the last inequality follows from Lemma 4.3.4(i). If there exists n such that the equality holds, then again Lemma 4.3.4(i) tells us that

$$\mathrm{Tree}(\gamma_n) = \mathrm{Tree}(\gamma) \qquad \mathrm{Tree}(\delta_n) = \mathrm{Tree}(\delta).$$

Therefore, by the induction hypothesis there exist $c'_n, d'_n \in \mathcal{D}_{N,\mathcal{A}}$, $\gamma', \delta' \in \mathrm{Rat}(E^\times N)$ such that $\bar{c} = c_n = c'_n t^n$, $\bar{d} = d_n = d'_n t^n$, $\gamma = \gamma' x^n$, $\delta = \delta' x^n$. Hence,

$$\begin{aligned}
\bar{b} &= \bar{c} + \bar{d} = (c'_n + d'_n) t^n \in \mathcal{D}_{N,\mathcal{A}} t^n \\
\beta &= \gamma + \delta = (\gamma' + \delta') x^n \in \mathrm{Rat}(E^\times N) x^n.
\end{aligned}$$

Case 2. If $\beta \in U \natural X \backslash (X \cup U)$, then there exist $\gamma, \delta \in U \natural X \backslash U$ such that $\beta = \gamma\delta$. By Lemma 4.3.7(vi),

$$\mathrm{Tree}(\gamma), \mathrm{Tree}(\delta) < \mathrm{Tree}(\beta).$$

Setting $c = \Phi(\gamma), d = \Phi(\delta)$, we obtain a decomposition $b = cd$. In particular, since b is non-zero, c and d must be non-zero. We claim that γ realizes the H-complexity of c, i.e., $\mathrm{Tree}_H(c) = \mathrm{Tree}(\gamma)$. Otherwise, there would exist (a non-zero) γ' with $\Phi(\gamma') = c$ satisfying $\mathrm{Tree}(\gamma') < \mathrm{Tree}(\gamma)$. Since γ', δ are non-zero, Lemma 4.3.7(i) tells us that $\mathrm{Tree}(\gamma'), \mathrm{Tree}(\delta) \neq 0_{\mathcal{T}}$, and hence using Lemma 4.3.7(v) and Lemma 4.3.4(ii)

$$\begin{aligned}
\mathrm{Tree}(\gamma'\delta) &= \mathrm{Tree}(\gamma')\mathrm{Tree}(\delta) < \mathrm{Tree}(\gamma)\mathrm{Tree}(\delta) \\
&= \mathrm{Tree}(\gamma\delta) = \mathrm{Tree}(\beta).
\end{aligned}$$

Since $\Phi(\gamma'\delta) = b$, this contradicts the minimality of β. Similarly, δ realizes the H-complexity of d, and therefore we have found a decomposition $b = cd$ with $\mathrm{Tree}_H(b) > \mathrm{Tree}_H(c), \mathrm{Tree}_H(d)$. Now, by the induction hypothesis, we can write $c = f(\bar{c})$, $d =$

$f(\bar{d})$ where $\bar{c} = \sum c_n$, $\bar{d} = \sum d_n$ with $c_n, d_n \in \mathcal{D}_{N,\mathcal{A}}t^n$, $\operatorname{Tree}_H(f(c_n)) \leq \operatorname{Tree}_H(c)$ and $\operatorname{Tree}_H(f(d_n)) \leq \operatorname{Tree}_H(d)$. Hence, we have an expression $\bar{b} = \bar{c}\bar{d} = \sum b_n$ with $b_n = \sum c_m d_{n-m}$ for every n, and $f(\bar{b}) = f(\bar{c})f(\bar{d}) = cd = b$. Since for $n < 0$ there are only finitely many non-zero coefficients c_n, d_n, the sum in the expression of b_n is finite. Let β_n, γ_n, δ_n be elements in $\operatorname{Rat}(E^\times H) \cup \{0\}$ realizing the H-complexities of $f(b_n), f(c_n), f(d_n)$, respectively, for all n. From the previous expression of b_n (and hence of $f(b_n)$) and the behavior of Tree_H with respect to sums and products we obtain

$$\operatorname{Tree}(\beta_n) \leq \sum \operatorname{Tree}(\gamma_m) \operatorname{Tree}(\delta_{n-m}).$$

Therefore, using Lemma 4.3.7,

$$\begin{aligned}
\log \operatorname{Tree}_H(f(b_n)) &= \log \operatorname{Tree}(\beta_n) \\
&\leq \log\left(\sum \operatorname{Tree}(\gamma_m) \operatorname{Tree}(\delta_{n-m})\right) \\
&\overset{(v)}{=} \log\left(\sum \operatorname{Tree}(\gamma_m \delta_{n-m})\right) \\
&\overset{(iii)}{=} \log\left(\operatorname{Tree}\left(\sum \gamma_m \delta_{n-m}\right)\right) \\
&\overset{(vii)}{=} \max\{\log(\operatorname{Tree}(\gamma_m \delta_{n-m}))\} \\
&\overset{(viii)}{=} \max\{\log \operatorname{Tree}(\gamma_m) + \log \operatorname{Tree}(\delta_{n-m})\} \\
&\overset{(*)}{\leq} \log \operatorname{Tree}(\gamma) + \log \operatorname{Tree}(\delta) \\
&\overset{(viii)}{=} \log \operatorname{Tree}(\gamma\delta) \\
&= \log \operatorname{Tree}(\beta) = \log \operatorname{Tree}_H(b).
\end{aligned}$$

To see $(*)$, note that since $\operatorname{Tree}(\gamma_m) \leq \operatorname{Tree}(\gamma)$, the definition of the order in $\mathcal{T}$ gives us $\log \operatorname{Tree}(\gamma_m) \leq \log \operatorname{Tree}(\gamma)$, and the corresponding inequality holds for δ. Thus, if γ_m, δ_{n-m} are non-zero, the result follows from Lemma 4.3.4(i), and otherwise we would have $\log \operatorname{Tree}(\gamma_m) + \log \operatorname{Tree}(\delta_{n-m}) = -\infty$, which is strictly less than any element in $\mathcal{T}$.

Now, if $\log \operatorname{Tree}_H(f(b_n)) < \log \operatorname{Tree}_H(b)$ for all n, then $\operatorname{Tree}_H(f(b_n)) < \operatorname{Tree}_H(b)$ for all n. Otherwise, if there exists n such that the equality holds, then by the previous expression there exists some integer m such that

$$\log \operatorname{Tree}(\gamma_m) + \log \operatorname{Tree}(\delta_{n-m}) = \log \operatorname{Tree}(\gamma) + \log \operatorname{Tree}(\delta).$$

Taking into account that γ and δ are non-zero, both summands in the right-hand side are at least $0_{\mathcal{T}}$, and hence the same applies to the left-hand side. Therefore, by Lemma 4.3.4(i) it must be the case that

$$\log \operatorname{Tree}(\gamma_m) = \log \operatorname{Tree}(\gamma) \qquad \log \operatorname{Tree}(\delta_{n-m}) = \log \operatorname{Tree}(\delta).$$

Since $\gamma, \delta \in U\natural X$, we have $\operatorname{width}(\gamma) = \operatorname{width}(\delta) = 1$ by Lemma 4.3.7(xiii), and consequently $\operatorname{Tree}(\gamma_m) \geq \operatorname{Tree}(\gamma)$ and $\operatorname{Tree}(\delta_{n-m}) \geq \operatorname{Tree}(\delta)$. Therefore, we have equality, and the induction hypothesis says that there exist $c'_m, d'_{n-m} \in \mathcal{D}_{N,\mathcal{A}}$, $\gamma', \delta' \in \mathrm{Rat}(E^\times N)$ such that $\bar{c} = c_m = c'_m t^m$, $\bar{d} = d'_{n-m} t^{n-m}$, $\gamma = \gamma' x^m$, $\delta = \delta' x^{n-m}$, and so

$$\bar{b} = \bar{c}\bar{d} = c'_m \tilde{\tau}^m (d'_{n-m}) t^n \in \mathcal{D}_{N,\mathcal{A}} t^n$$
$$\beta = \gamma\delta = \gamma' \bar{\tau}^m (\delta') x^n \in \mathrm{Rat}(E^\times N) x^n.$$

Case 3. If $\beta \in X$, then there exists $\gamma \in \mathbb{N}[U \natural X] \setminus \{0\}$ such that $\beta = \gamma^\diamond$. Since γ is non-zero, Lemma 4.3.7(xii) tells us that $\mathrm{Tree}(\gamma) < \mathrm{Tree}(\beta)$, and setting $c = \Phi(\gamma) \in \mathcal{D}_{H,\mathcal{P}}$ we obtain that

$$b = \Phi(\gamma^\diamond) = c^\diamond.$$

Moreover, since b is non-zero, the definition of the $\diamond$-map in $\mathcal{D}_{H,\mathcal{P}}$ implies that $c^\diamond = c^{-1}$ (in particular c is non-zero). We claim that γ realizes the H-complexity of c, i.e., $\mathrm{Tree}_H(c) = \mathrm{Tree}(\gamma)$. Otherwise, there would exist (a non-zero) γ' with $\Phi(\gamma') = c$ satisfying $\mathrm{Tree}(\gamma') < \mathrm{Tree}(\gamma)$, from where using Lemma 4.3.7(xi) we would get

$$\begin{aligned} \mathrm{Tree}((\gamma')^\diamond) &= \exp^2 \mathrm{Tree}(\gamma') \\ &< \exp^2 \mathrm{Tree}(\gamma) \\ &= \mathrm{Tree}(\gamma^\diamond) = \mathrm{Tree}(\beta). \end{aligned}$$

The inequality follows from the way the order is defined in $\mathcal{T}$, since comparing those trees amounts to compare $\mathrm{Tree}(\gamma')$ and $\mathrm{Tree}(\gamma)$. Since $\Phi((\gamma')^\diamond) = \Phi(\gamma')^\diamond = c^{-1} = b$, this contradicts the minimality of β. Hence, we have $b = c^{-1}$ with $\mathrm{Tree}_H(c) < \mathrm{Tree}_H(b)$. Now, by the induction hypothesis, we can write $c = f(\bar{c})$ where $\bar{c} = \sum c_n$ with $c_n \in \mathcal{D}_{N,\mathcal{A}} t^n$ and $\mathrm{Tree}_H(f(c_n)) \leq \mathrm{Tree}_H(c)$ for every n. It is important to notice also that

$$\mathrm{Tree}_H(f(c_n)) \leq \mathrm{Tree}_H(c) < \mathrm{Tree}_H(b) \leq \mathrm{Tree}_H(a).$$

Moreover, assume that $c_n = c'_n t^n$ for some $c'_n \in \mathcal{D}_{N,\mathcal{A}}$. By Remark 4.3.12(2.), $f(\mathcal{D}_{N,\mathcal{A}}) = \mathcal{D}_{N,\mathcal{P}}$, so that $f(c'_n) \in \mathcal{D}_{N,\mathcal{P}} \subseteq \mathcal{D}_{H,\mathcal{P}}$, and we also have $f(t^n) = f\tilde{\phi}(x^n) \in f\tilde{\phi}(E * H) \subseteq \mathcal{D}_{H,\mathcal{P}}$. Thus, $f(c_n) \in \mathcal{D}_{H,\mathcal{P}}$ has strictly less H-complexity than a, and hence the hypothesis of the proposition tells us that all non-zero $f(c_n)$ are invertible in $\mathcal{D}_{H,\mathcal{P}}$. Since t is invertible, then we deduce that $f(c'_n) = f(c_n) f(t^{-n})$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$ and hence in $\mathcal{D}_{N,\mathcal{P}}$, and again since f restricts to an isomorphism from $\mathcal{D}_{N,\mathcal{A}}$ to $\mathcal{D}_{N,\mathcal{P}}$, this implies that c'_n is invertible in $\mathcal{D}_{N,\mathcal{A}}$.

In particular, for $k = \min\{n : c_n \neq 0\}$ (the least element in the support of $\bar{c}$), c'_k is invertible, what implies that $\bar{c}$ is invertible in $\mathcal{D}_{N,\mathcal{A}}((t,\tilde{\tau}))$ (cf. [Sán08, Examples 1.6(c),(d), and Examples 1.43(e)]). Therefore, if $\bar{b} = \sum b_n$ denotes the inverse of $\bar{c} = \sum c_n$ in $\mathcal{D}_{N,\mathcal{A}}((t;\tilde{\tau}))$, then

$$f(\bar{b}) = f(\bar{c}^{-1}) = f(\bar{c})^{-1} = c^{-1} = b.$$

Taking a deeper look to the construction of $\bar{c}^{-1}$, we can see that b_n is constructed using sums and products of the elements c_k^{-1} and $-c_m$, for $m \in C_n = \{k+1, \ldots, 2k+n\}$. Let $\beta_n, \gamma_n \in \mathrm{Rat}(E^\times H) \cup \{0\}$ realizing the H-complexities of $f(b_n)$ and $-f(c_n)$ for every n.

Note first that $-f(c_k)$, and therefore γ_k, is non-zero. Thus $\delta_k = (-1)\gamma_k$ is non-zero and $\Phi(\delta_k) = f\tilde{\phi}(-1)\Phi(\gamma_k) = f(c_k)$ because Φ is a morphism of $E^\times H$-semirings and $\tilde{\phi}$ the restriction of a ring homomorphism (what implies that $\tilde{\phi}(-1) = -1$). Since $\mathrm{Tree}(\delta_k) = \mathrm{Tree}(\gamma_k)$ by Lemma 4.3.7(v) and (ii), and $\Phi(\delta_k^\diamond) = \Phi(\delta_k)^\diamond = f(c_k)^\diamond = f(c_k)^{-1}$ because Φ preserves $\diamond$, we obtain from the rationality of Tree on non-zero elements that

$$\mathrm{Tree}_H(f(c_k)^{-1}) \leq \mathrm{Tree}(\delta_k^\diamond) = \mathrm{Tree}(\delta_k)^\diamond = \mathrm{Tree}(\gamma_k)^\diamond = \mathrm{Tree}(\gamma_k^\diamond). \tag{4.5}$$

Secondly, as $\mathrm{Tree}_H(f(c_k)) = \mathrm{Tree}_H(-f(c_k))$, we have by Lemma 4.3.7(xii),

$$\begin{aligned} \log^2 \mathrm{Tree}(\gamma_k^\diamond) &\overset{(xii)}{=} \mathrm{Tree}(\gamma_k) = \mathrm{Tree}_H(-f(c_k)) \\ &= \mathrm{Tree}_H(f(c_k)) \\ &\leq \mathrm{Tree}_H(c) = \mathrm{Tree}(\gamma) \\ &\overset{(xii)}{=} \log^2 \mathrm{Tree}(\gamma^\diamond) = \log^2 \mathrm{Tree}(\beta). \end{aligned} \tag{4.6}$$

Similarly, for every $m > k$, we have

$$\begin{aligned} \log^2 \mathrm{Tree}(\gamma_m) &< \mathrm{Tree}(\gamma_m) = \mathrm{Tree}_H(-f(c_m)) \\ &= \mathrm{Tree}_H(f(c_m)) \\ &\leq \mathrm{Tree}_H(c) = \mathrm{Tree}(\gamma) \\ &\overset{(xii)}{=} \log^2 \mathrm{Tree}(\gamma^\diamond) = \log^2 \mathrm{Tree}(\beta), \end{aligned} \tag{4.7}$$

where the strict inequality follows because the first expression is either $-\infty$, and hence strictly less than any element in $\mathcal{T}$, or its height is strictly lower than the height of $\mathrm{Tree}(\gamma_m)$ (and the ordering in $\mathcal{T}$ refines the order by height).

In the third place, the expression of b_n in terms of c_k^{-1} and $-c_m$ (using sums and products) gives us the corresponding expression of $f(b_n)$ in terms of $f(c_k)^{-1}$ and $-f(c_m)$. Taking into account the behavior of Tree_H with respect to sums and products, we deduce that $\mathrm{Tree}_H(f(b_n))$ is less or equal than the same expression in terms of $\mathrm{Tree}_H(f(c_k)^{-1})$ and $\mathrm{Tree}_H(-f(c_m))$, and hence by Eq. (4.5) and Lemma 4.3.4, less or equal than the same expression in terms of $\mathrm{Tree}(\gamma_k^\diamond)$ and $\mathrm{Tree}(\gamma_m)$. Thus, $\log^2 \mathrm{Tree}_H(f(b_n))$ is less or equal than $\log^2$ of the latter expression (in general, if $X \leq Y$ in $\mathcal{T}$, then $\log^2 X \leq \log^2 Y$, because otherwise the condition $\log^2 X > \log^2 Y$ allows us to determine that $\log X > \log Y$ and hence that $X > Y$). Therefore, using repeatedly Lemma 4.3.7(ix) and (x), and Eqs. (4.6) and (4.7),

$$\begin{aligned} \log^2 \mathrm{Tree}(\beta_n) &= \log^2 \mathrm{Tree}_H(f(b_n)) \\ &\leq \max \left\{ \log^2 \mathrm{Tree}(\gamma_k^\diamond), \max_{m \in C_n} \{ \log^2 \mathrm{Tree}(\gamma_m) \right\} \\ &\leq \log^2 \mathrm{Tree}(\beta). \end{aligned}$$

If for every n, $\log^2 \mathrm{Tree}(\beta_n) < \log^2 \mathrm{Tree}(\beta)$, then we conclude that for every n, $\mathrm{Tree}(\beta_n) < \mathrm{Tree}(\beta)$ (see the reasoning in the precedent paragraph). If equality holds for some n, then since the inequality in (4.7) is strict, we obtain from (4.6) that $\mathrm{Tree}_H(f(c_k)) = \mathrm{Tree}_H(c) = \mathrm{Tree}(\gamma)$. The induction hypothesis then tells us that there exist $c'_k \in \mathcal{D}_{N,\mathcal{A}}$, $\gamma' \in \mathrm{Rat}(E^\times N)$ such that $\bar{c} = c_k = c'_k t^k$, $\gamma = \gamma' x^k$ and so

$$\bar{b} = \bar{c}^{-1} = t^{-k}(c'_k)^{-1} = \tilde{\tau}^{-k}(c'_k)^{-1}t^{-k} \in \mathcal{D}_{N,\mathcal{A}}t^{-k}$$
$$\beta = \gamma^\diamond = x^{-k}(\gamma')^\diamond = \bar{\tau}^{-k}((\gamma')^\diamond)x^{-k} \in \mathrm{Rat}(E^\times N)x^{-k}.$$

In each of the different cases we have seen that $b = f(\bar{b})$ for some element $\bar{b} = \sum b_n \in \mathcal{D}_{N,\mathcal{A}}((t;\tilde{\tau}))$ with $\mathrm{Tree}_H(f(b_n)) \leq \mathrm{Tree}_H(b)$ for every n, and moreover that if equality holds for some n then $\bar{b} = b_n \in \mathcal{D}_{N,\mathcal{A}}t^n$ and the fixed arbitrary β realizing the H-complexity of b lies in $\mathrm{Rat}(E^\times N)x^n$.

We want to show that in the latter case any other β' realizing the H-complexity of b also lies in $\mathrm{Rat}(E^\times N)x^n$. Indeed, the same reasoning gives us an expression $\beta' = \gamma' + \delta'$, $\beta' = \gamma'\delta'$ or $\beta' = \gamma'^\diamond$, and hence the corresponding expressions $b = c_0 + d_0$, $b = c_0 d_0$ or $b = c_0^{-1}$, where γ' (resp. δ') realizes the H-complexity of c_0 (resp. d_0) and it is strictly less complex than β'. Thus, by induction we have $c_0 = f(\tilde{c})$, $d_0 = f(\tilde{d})$ such that $\tilde{c} = \sum \tilde{c}_k$, $\tilde{d} = \sum \tilde{d}_k$ with $\mathrm{Tree}_H(f(\tilde{c}_k)) \leq \mathrm{Tree}_H(c_0)$ and $\mathrm{Tree}_H(f(\tilde{d}_k)) \leq \mathrm{Tree}_H(d_0)$. From here, we obtain an element $\tilde{b} \in \mathcal{D}_{N,\mathcal{A}}((t,\tilde{\tau}))$ such that $f(\tilde{b}) = b$. The injectivity of f shows that necessarily $\tilde{b} = \bar{b} = b_n \in \mathcal{D}_{N,\mathcal{A}}t^n$. In particular, we have that $\mathrm{Tree}_H(b) = \mathrm{Tree}_H(f(b_n))$, and following the reasoning in each case we deduce that $\tilde{c}$ and $\tilde{d}$ are monomials such that $\tilde{b} \in \mathcal{D}_{N,\mathcal{A}}t^n$. Thus, γ', δ' lie in $\mathrm{Rat}(E^\times H)\langle x\rangle$ and the corresponding operation shows that $\beta' \in \mathrm{Rat}(E^\times H)x^n$ for the same n.

Since the converse always holds, i.e., if $b = f(b_n)$, then $\mathrm{Tree}_H(b) = \mathrm{Tree}_H(f(b_n))$, the proof is finished. $\square$

As a first application of the previous proposition we give an alternative proof of Hughes' theorem Theorem 3.4.23.

Theorem 4.3.14. *Let $E * G$ be a crossed product of a division ring E and a locally indicable group G. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two Hughes-free division $E * G$-rings of fractions, then there exists a unique $E * G$-isomorphism $\varphi : \mathcal{D}_1 \to \mathcal{D}_2$.*

Proof. Let us denote by $\phi_i : E * G \hookrightarrow \mathcal{D}_i$ the (Hughes-free) embedding of $E * G$ in $\mathcal{D}_i$. Set $\mathcal{S} = \mathcal{D}_1 \times \mathcal{D}_2$, $\phi = (\phi_1, \phi_2) : E * G \to \mathcal{S}$ and $\mathcal{D} = \mathcal{D}_{G,\mathcal{S}}$ the division closure of $\phi(E * G)$ in $\mathcal{S}$. For every subgroup H of G, let $\mathcal{D}_{H,\mathcal{S}}$ (resp. $\mathcal{D}_{H,\mathcal{D}}$, $\mathcal{D}_{H,1}$, $\mathcal{D}_{H,2}$) denote the division closure of the corresponding image of $E * H$ in $\mathcal{S}$ (resp. in $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$). Denote by $\pi_i : \mathcal{D} \to \mathcal{D}_i$ $(i = 1, 2)$ the restriction to $\mathcal{D}$ of the canonical projections, and let $\Phi : \mathrm{Rat}(E^\times G) \to \mathcal{D}$ be the unique morphism of rational $E^\times G$-rings. By induction on the G-complexity Tree_G we will show that any non-zero element $a \in \mathcal{D}$ is invertible.

If $\mathrm{Tree}_G(a) = 1_{\mathcal{T}}$ and α realizes the G-complexity of a, then by Lemma 4.3.7(ii) $\alpha \in E^\times G$, and therefore $\Phi(a) = \phi(a) \in \phi(E^\times G)$ is invertible.

Now assume that $\mathrm{Tree}_G(a) > 1_{\mathcal{T}}$ and that for every $0 \neq b \in \mathcal{D}$ such that $\mathrm{Tree}_G(b) < \mathrm{Tree}_G(a)$, b is invertible. Let $\alpha \in \mathrm{Rat}(E^\times G)$ realize the G-complexity of a. Using Theorem 4.3.8, there exists a finitely generated subgroup $\mathrm{source}(\alpha)$ of $E^\times G$ such that $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha))E^\times G$. Observe that if $\alpha = pu$ for some primitive p and $u \in E^\times G$, then $\mathrm{Tree}(\alpha) = \mathrm{Tree}(p)$ by Lemma 4.3.7(v) and (ii), and $a = \Phi(p)\phi(u)$. Therefore, $a' = a\phi(u^{-1})$ is an element of $\mathcal{D}$ which is invertible if and only if a is invertible, and p realizes its G-complexity because $\Phi(p) = a'$ and $\mathrm{Tree}_G(a') = \mathrm{Tree}_G(a) = \mathrm{Tree}(p)$. For this reason we can assume without loss of generality that α is already primitive, what implies that $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha))$.

Consider the canonical map $\pi_G : E^\times G \to E^\times G/E^\times \cong G$ and let H be the image of $\mathrm{source}(\alpha)$, i.e., $H = \pi_G(\mathrm{source}(\alpha))$, which is a finitely generated subgroup of G. By construction we have $\mathrm{source}(\alpha) \leq E^\times H$, and hence by Proposition 4.3.9 we deduce that $a = \Phi(\alpha) \in \mathcal{D}_{H,\mathcal{S}}$. Another consequence is that $\alpha \in \mathrm{Rat}(E^\times H)$, and hence

$$\mathrm{Tree}_H(a) = \mathrm{Tree}_G(a) = \mathrm{Tree}(\alpha).$$

If $H = \{e\}$ is trivial, then $\phi(E * H) = \phi(E)$ is a division subring of $\mathcal{S}$ and hence $a \in \mathcal{D}_{H,\mathcal{S}} = \phi(E)$ is invertible. Otherwise, there exists a normal subgroup $N \triangleleft H$ such that H/N is infinite cyclic. Let $x \in E^\times H$ be such that $H/N = \langle N\pi_H(x)\rangle$, let τ denote the automorphism of $E*N$ induced by left conjugation by x and $\tilde{\tau}_i$ denote the automorphism of $\mathcal{D}_{N,i}$ given by left conjugation by $\phi_i(x)$. Then $\tilde{\tau}_i \circ \phi_i = \phi_i \circ \tau$ and, as in the the proof of the "only if" part in Proposition 3.4.31 we obtain from the Hughes-freeness of $\mathcal{D}_i$ that there exists an $E * H$-isomorphism $\mathcal{D}_{H,i} \cong \mathcal{D}_{N,i}(t; \tilde{\tau}_i)$ that acts as the identity on $\mathcal{D}_{N,i}$ and sends $\phi_i(x) \mapsto t$. This gives us an embedding $\psi_i : \mathcal{D}_{H,i} \to \mathcal{D}_{N,i}((t, \tilde{\tau}_i))$ for $i = 1, 2$.

We want to apply Proposition 4.3.13, and for that we need to show that conditions (i), (ii) and (iii) in its statement are satisfied.

(i) Set $\mathcal{A} = \mathcal{D}_{N,1} \times \mathcal{D}_{N,2}$ and observe that $\mathcal{A}$ is regular because $\mathcal{D}_{N,i}$ is a division ring (hence regular) for $i = 1, 2$, and a product of regular rings is again regular. Moreover, $\phi(E*N) \subseteq \phi_1(E*N) \times \phi_2(E*N) \subseteq \mathcal{A}$, so that $(\mathcal{A}, \phi)$ satisfies condition (i).

(ii) Since $\tilde{\tau}_i$ is an automorphism of $\mathcal{D}_{N,i}$ for every i, then $\tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}_2) : \mathcal{A} \to \mathcal{A}$ is an automorphism of $\mathcal{A}$, and for every $y \in E * N$,

$$\begin{aligned} \tilde{\tau} \circ \phi(y) &= \tilde{\tau}(\phi_1(y), \phi_2(y)) = (\tilde{\tau}_1\phi_1(y), \tilde{\tau}_2\phi_2(y)) \\ &= (\phi_1\tau(y), \phi_2\tau(y)) = \phi \circ \tau(y), \end{aligned}$$

from where $\tilde{\tau} \circ \phi = \phi \circ \tau$ and condition (ii) is satisfied.

(iii) Here, we directly take $\mathcal{P} = \mathcal{A}((t; \tilde{\tau}))$, so that (iii) is satisfied and $\mathcal{D}_{N,\mathcal{P}} = \mathcal{D}_{N,\mathcal{A}}$.

Set $\mathcal{B} = \mathcal{D}_{H,1} \times \mathcal{D}_{H,2}$, which as before is a regular ring containing $\phi(E * H)$, and put $\psi = (\psi_1, \psi_2) : \mathcal{B} \to \mathcal{D}_{N,1}((t; \tilde{\tau}_1)) \times \mathcal{D}_{N,2}((t; \tilde{\tau}_2))$, which is injective since ψ_1 and ψ_2 are injective. Let $\psi' : \mathcal{D}_{N,1}((t; \tilde{\tau}_1)) \times \mathcal{D}_{N,2}((t; \tilde{\tau}_2)) \to \mathcal{P}$ denote the isomorphism given by $(\sum_k y_k t^k, \sum_k z_k t^k) \mapsto \sum_k (y_k, z_k) t^k$. An important observation here is that the map $\tilde{\phi} : E * H \to \mathcal{P}$ contructed for the proof of Proposition 4.3.13 factorizes through

$$E * H \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{D}_{N,1}((t; \tilde{\tau}_1)) \times \mathcal{D}_{N,2}((t; \tilde{\tau}_2)) \xrightarrow{\psi'} \mathcal{P}$$

Indeed, since ψ_i acts as the identity on $\mathcal{D}_{N,i}$, we have that for every $y \in E * N$,

$$\psi' \circ \psi \circ \phi(y) = \psi'(\phi_1(y), \phi_2(y)) = (\phi_1(y), \phi_2(y)) = \tilde{\phi}(y) \in \mathcal{A}$$

and

$$\begin{aligned} \psi' \circ \psi \circ \phi(x) &= \psi'\psi(\phi_1(x), \phi_2(x)) = \psi'(\psi_1\phi_1(x), \psi_1\phi_2(x)) \\ &= \psi'(t,t) = t. \end{aligned}$$

Since this completely determines the image of any other element of $E * H$, we deduce that $\tilde{\phi} = \psi' \circ \psi \circ \phi$.

The reason why this is important is the following. Since $\tilde{\psi} = \psi' \circ \psi$ is injective and $\tilde{\psi}(\mathcal{B})$ is regular containing the image of $E * H$, we have by Lemma 3.3.3(ii) and (iii) that

$$\mathcal{D}_{H,\mathcal{S}} = \mathcal{D}_{H,\mathcal{B}} \overset{\tilde{\psi}}{\cong} \mathcal{D}_{H,\tilde{\psi}(\mathcal{B})} = \mathcal{D}_{H,\mathcal{P}}$$

where the isomorphism is of $E * H$-rings. Therefore, Lemma 4.3.11 tells us that for every $y \in \mathcal{D}_{H,\mathcal{S}}$, $\mathrm{Tree}_H(y) = \mathrm{Tree}_H(\tilde{\psi}(y))$. Moreover, if $a' = \tilde{\psi}(a)$ the cited lemma shows that α also realizes the H-complexity of a'.

Observe now that for every $0 \neq b' \in \mathcal{D}_{H,\mathcal{P}}$, there exists $b \in \mathcal{D}_{H,\mathcal{S}}$ with $\tilde{\psi}(b) = b'$ and $\mathrm{Tree}_H(b) = \mathrm{Tree}_H(b')$. Thus, if $\mathrm{Tree}_H(b') < \mathrm{Tree}_H(a')$, then

$$\mathrm{Tree}_G(b) \leq \mathrm{Tree}_H(b) = \mathrm{Tree}_H(b') < \mathrm{Tree}_H(a') = \mathrm{Tree}_H(a) = \mathrm{Tree}_G(a).$$

Consequently, the induction hypothesis implies that b is invertible in $\mathcal{D}$ and hence in $\mathcal{D}_{H,\mathcal{B}}$, and thus $b' = \tilde{\psi}(b)$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$. In other words, every element in $\mathcal{D}_{H,\mathcal{P}}$ with less H-complexity than a' is invertible. Hence, Proposition 4.3.13 applies and tells us that $a' \in \mathcal{D}_{N,\mathcal{P}}((t;\tilde{\tau}))$ and $a' = \sum_k a_k$ with $\mathrm{Tree}_H(a_k) \leq \mathrm{Tree}_H(a')$. Moreover, we claim that there are at least two non-zero summands. Otherwise, if $a' = a_n$, the same proposition tells us that $\alpha \in \mathrm{Rat}(E^\times N)x^n \subseteq \mathrm{Rat}(E^\times N)E^\times H$, and then Theorem 4.3.8(iv) states that $\mathrm{source}(\alpha) \leq E^\times N$, what implies that $H \leq N$, a contradiction.

Hence, $\mathrm{Tree}_H(a_k) < \mathrm{Tree}_H(a')$ for all k. In particular, if n is the smallest k such that a_k is non-zero, we deduce as before that the element $a_n \in \mathcal{D}_{N,\mathcal{P}}t^n$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$. This implies that a' is invertible in $\mathcal{D}_{N,\mathcal{P}}((t;\tilde{\tau})) \subseteq \mathcal{P}$, and hence in $\mathcal{D}_{H,\mathcal{P}}$. Therefore $a = \tilde{\psi}^{-1}(a')$ is a non-zero element of $\mathcal{D}$ which is invertible in $\mathcal{D}_{H,\mathcal{S}} \subseteq \mathcal{S}$, and hence in $\mathcal{D}$, as we wanted to show.

We have proved that $\mathcal{D}$ is a division ring, what implies that $\pi_i : \mathcal{D} \to \mathcal{D}_i$ is injective for $i = 1, 2$. Moreover, we have a commutative diagram

$$\begin{tikzcd} E * G \arrow[r, "\phi"] \arrow[d, "\phi_i"'] & \mathcal{D} \arrow[d, "\pi_i"] \\ \mathcal{D}_i \arrow[r, "\mathrm{id}_{\mathcal{D}_i}"] & \mathcal{D}_i, \end{tikzcd}$$

where $\mathcal{D}, \mathcal{D}_i$ are division rings (hence regular) and ϕ, ϕ_i are epic (Proposition 3.1.13) Thus, by Corollary 4.1.15, $\pi_i(\mathcal{D}) = \mathrm{id}_{\mathcal{D}_i}(\mathcal{D}_i) = \mathcal{D}_i$, and hence π_i is an isomorphism.

Since the following also commutes,

$$\begin{tikzcd} & E * G \arrow[dl, "\phi_1"'] \arrow[d, "\phi"] \arrow[dr, "\phi_2"] & \\ \mathcal{D}_1 \arrow[r, "\pi_1^{-1}"'] & \mathcal{D} \arrow[r, "\pi_2"'] & \mathcal{D}_2 \end{tikzcd}$$

we have that $\varphi = \pi_2 \circ \pi_1^{-1}$ is an $E * G$-isomorphism between $\mathcal{D}_1$ and $\mathcal{D}_2$. The uniqueness comes from epicity, because if φ' is another $E * G$-isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$, then

$$\begin{tikzcd} & E * G \arrow[dl, "\phi_1"'] \arrow[d, "\phi_2"] \arrow[dr, "\phi_1"] & \\ \mathcal{D}_1 \arrow[r, "\varphi"'] & \mathcal{D}_2 & \mathcal{D}_1 \arrow[l, "\varphi'"] \end{tikzcd}$$

commutes and hence $\varphi' \circ \phi_1 = \phi_2 = \varphi \circ \phi_1$. Since ϕ_1 is epic, $\varphi' = \varphi$. $\square$

Unlike the proof presented in [JL20, Theorem 5.2], we did not want to skip here any detail because it represents the general strategy of proof using Proposition 4.3.13. This makes notation harder and make some "identifications" turn into isomorphisms, but we hope it helps the reader to understand better how the argument works. In future applications of Proposition 4.3.13 we may just direct the reader to the proof of this theorem to unravel the details.

4.3.4 Epic $*$-regular $K[G]$-rings

The next and last example of rational semiring was central in a previous version of [JL20]. Here, G is a group, K is a subfield of $\mathbb{C}$ closed under complex conjugation and $K[G]$ is the group ring with the proper involution $*$ defined in Section 4.2, i.e., the one taking $\sum_{g\in G} a_g g$ to the element $\sum_{g\in G} \bar{a}_g g^{-1}$. Since K is closed under complex conjugation, the group ring is $*$-closed.

Lemma 4.3.15. *In the previous setting, if $K[G]$ is a $*$-subring of a $*$-regular ring $\mathcal{U}$ and $K[G] \hookrightarrow \mathcal{U}$ is epic, then $\mathcal{U}$ is a rational $K^\times G$-semiring with $\diamond$-operation given by taking relative inverses.*

Proof. The given ring structure of $\mathcal{U}$ together with the usual multiplication by elements of $K^\times G$ make $\mathcal{U}$ a $K^\times G$-semiring. We need to prove that the operation $\square^{[-1]}$ is compatible with this structure, i.e., that for every $u, v \in K^\times G$, $x \in \mathcal{U}$ we have

$$(uxv)^{[-1]} = v^{-1}x^{[-1]}u^{-1}.$$

Set $e = \mathrm{RP}(x)$, $f = \mathrm{LP}(x)$, $y = uxv$, $z = v^{-1}x^{[-1]}u^{-1}$. By Remark 4.1.3 we just need to check that yz and zy are projections and that $yzy = y$, $zyz = z$.

Observe that if $u = ag$ for some non-zero $a \in K$, then $u^{-1} = a^{-1}g^{-1}$. Since the embedding is epic and $K \subseteq Z(K[G])$, a commutes with every element in $\mathcal{U}$ by Corollary 4.1.12 and, therefore, $yz = ufu^{-1} = gfg^{-1}$ is a projection. Similarly, zy is a projection. Finally, $yzy = uxx^{[-1]}xv = uxv = y$, and analogously $zyz = z$, what shows the desired result. $\square$

Recall from Proposition 4.1.16 that the conditions required in the lemma are equivalent to saying that $\mathcal{U}$ is the $*$-regular closure of $K[G]$ in $\mathcal{U}$. In particular, the above operation endows $R_{K[G]}$, the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$ with a structure of $K^{\times}G$-semiring.

Similar considerations to those of Section 4.3.3, such as Proposition 4.3.9, the notion of complexity or a specialized version of Proposition 4.3.13 can also be developed here.

4.4 The strong Atiyah conjecture for locally indicable groups

In this section we finally show that locally indicable groups satisfy the Atiyah conjecture over $\mathbb{C}$, and hence over any other subfield of $\mathbb{C}$ in view of Proposition 4.2.5(5.).

The general situation is the one described in Section 4.1.1, i.e., G is a locally indicable group, K a subfield of $\mathbb{C}$ closed under complex conjugation, $K[G]$ is endowed with the usual involution $*$ and there exists a rank function rk on $K[G]$ such that

1. rk is $*$-regular and its $*$-regular envelope $(\mathcal{U}, \mathrm{rk}', \phi)$ is positive definite.

2. rk is a Hughes-free Sylvester matrix rank function on $K[G]$.

Let H be a non-trivial finitely generated subgroup of G, $N \triangleleft H$ such that H/N is infinite cyclic, $x \in H$ such that $H/N = \langle Nx \rangle$ and τ the automorphism of $K[N]$ induced by left conjugation by x. Under the previous conditions we constructed a $*$-automorphism $\tilde{\tau}$ of $\mathcal{U}_N$ such that $\tilde{\tau} \circ \phi = \phi \circ \tau$, a $*$-regular ring $\mathcal{P} = \mathcal{P}^{\mathcal{U}_N}_{\omega,\tilde{\tau}}$ (ω a non-principal ultrafilter on $\mathbb{N}$) and a commutative diagram Eq. (4.4)

$$\begin{tikzcd}
K[N] \arrow[r, hook] \arrow[d, "\phi"] & K[H] \arrow[r, "\phi"] \arrow[d, "\tilde{\phi}"] & \mathcal{U}_H \arrow[d, "\varphi"] \\
\mathcal{U}_N \arrow[r, hook] & \mathcal{U}_N((t;\tilde{\tau})) \arrow[r, "f_\omega"'] & \mathcal{P}
\end{tikzcd}$$

where $\mathcal{U}_N$ and $\mathcal{U}_H$ denote, respectively, the $*$-regular closures of $\phi(K[N])$ and $\phi(K[H])$ in $\mathcal{U}$, f_ω and φ are injective, and $\tilde{\phi}$ (previously $\tilde{\phi} \circ \iota$) is the map acting as ϕ on $K[N]$ and sending $x \mapsto t$. From its commutativity we can observe:

a.) $(\mathcal{A} = \mathcal{U}_N, \phi)$, $\tilde{\tau}$ and $(\mathcal{P}, f_\omega)$ satisfy the conditions (i), (ii) and (iii) needed in order to apply Proposition 4.3.13, and $\tilde{\phi}$ is the actual $\tilde{\phi}$ constructed in Remark 4.3.12(1.).

b.) Let $\mathcal{D}_{H,\mathcal{U}}$ and $\mathcal{D}_{H,\mathcal{P}}$ denote, respectively, the division closure of $\phi(K[H])$ and $f_\omega\tilde{\phi}(K[H])$ in $\mathcal{U}$ and $\mathcal{P}$. Since $\mathcal{U}_H$ is regular and φ is an embedding, $\varphi(\mathcal{U}_H)$ is regular and by Lemma 3.3.3(2) and (3) we have that φ restricts to an isomorphism of $K[H]$-rings from $\mathcal{D}_{H,\mathcal{U}_H}$ to $\mathcal{D}_{H,\varphi(\mathcal{U}_H)}$ and that

$$\mathcal{D}_{H,\mathcal{U}} = \mathcal{D}_{H,\mathcal{U}_H} \cong \mathcal{D}_{H,\varphi(\mathcal{U}_H)} = \mathcal{D}_{H,\mathcal{P}}.$$

This shall be used for the inductive step in the main result.

Theorem 4.4.1. *Let G be a locally indicable group, K a subfield of $\mathbb{C}$ closed under complex conjugation. Let* rk *be a $*$-regular Hughes-free Sylvester matrix rank function on $K[G]$ with positive definite $*$-regular envelope $(\mathcal{U}, \mathrm{rk}', \phi)$. Then $\mathcal{U}$ is a division ring.*

Proof. Let $\mathcal{D} = \mathcal{D}_{G,\mathcal{U}}$ be the division closure of $\phi(K[G])$ in $\mathcal{U}$, and for any subgroup $H \leq G$, denote by $\mathcal{D}_{H,\mathcal{U}}$ and $\mathcal{U}_H$ the division and the $*$-regular closures of $\phi(K[H])$, respectively, in $\mathcal{U}$. Consider the universal morphism of rational $K^\times G$-semirings $\Phi :$ $\mathrm{Rat}(K^\times G) \to \mathcal{D}$. By induction on the G-complexity Tree_G we will show that any non-zero element $a \in \mathcal{D}$ is invertible.

If $\mathrm{Tree}_G(a) = 1_\mathcal{T}$ and α realizes the G-complexity of a, then by Lemma 4.3.7(ii) $\alpha \in K^\times G$, and therefore $\Phi(a) = \phi(a) \in \phi(K^\times G)$ is invertible.

Now assume that $\mathrm{Tree}_G(a) > 1_\mathcal{T}$ and that for every $0 \neq b \in \mathcal{D}$ such that $\mathrm{Tree}_G(b) <$ $\mathrm{Tree}_G(a)$, b is invertible. Let $\alpha \in \mathrm{Rat}(K^\times G)$ realize the G-complexity of a. As in the proof of Theorem 4.3.14, we can assume without loss of generality that α is primitive, so that $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha))$. Consider the map $\pi_G : K^\times G \to K^\times G/K^\times \cong G$ and set $H = \pi_G(\mathrm{source}(\alpha))$, which is a finitely generated subgroup of G. By construction we have $\mathrm{source}(\alpha) \leq K^\times H$, and hence by Proposition 4.3.9 we deduce that $a = \Phi(\alpha) \in \mathcal{D}_{H,\mathcal{U}} = \mathcal{D}_{H,\mathcal{U}_H}$. Another consequence is that $\alpha \in \mathrm{Rat}(K^\times H)$, and hence

$$\mathrm{Tree}_H(a) = \mathrm{Tree}_G(a) = \mathrm{Tree}(\alpha).$$

If $H = \{e\}$ is trivial, then $\phi(K[H]) = \phi(K)$ is a division subring of $\mathcal{U}$ and hence $a \in \mathcal{D}_{H,\mathcal{U}} = \phi(K)$ is invertible. Otherwise, there exists a normal subgroup $N \triangleleft H$ such that H/N is infinite cyclic. Let $x \in H$ be such that $H/N = \langle Nx \rangle$ and let τ denote the automorphism of $K[N]$ induced by left conjugation by x. Construct the previous diagram and set $\mathcal{A}$ and $\mathcal{P}$ as in a.). As observed in b.), one has

$$\mathcal{D}_{H,\mathcal{U}} = \mathcal{D}_{H,\mathcal{U}_H} \overset{\varphi}{\cong} \mathcal{D}_{H,\varphi(\mathcal{U}_H)} = \mathcal{D}_{H,\mathcal{P}}$$

where the isomorphism is of $K[H]$-rings. Therefore, Lemma 4.3.11 tells us that for every $y \in \mathcal{D}_{H,\mathcal{U}}$, $\mathrm{Tree}_H(y) = \mathrm{Tree}_H(\varphi(y))$. Moreover, if $a' = \varphi(a)$ the cited lemma shows that α also realizes the H-complexity of a'.

Observe now that for every $0 \neq b' \in \mathcal{D}_{H,\mathcal{P}}$, there exists $b \in \mathcal{D}_{H,\mathcal{U}}$ with $\varphi(b) = b'$ and $\mathrm{Tree}_H(b) = \mathrm{Tree}_H(b')$. Thus, if $\mathrm{Tree}_H(b') < \mathrm{Tree}_H(a')$, then

$$\mathrm{Tree}_G(b) \leq \mathrm{Tree}_H(b) = \mathrm{Tree}_H(b') < \mathrm{Tree}_H(a') = \mathrm{Tree}_H(a) = \mathrm{Tree}_G(a).$$

Consequently, the induction hypothesis implies that b is invertible in $\mathcal{D}$ and hence in $\mathcal{D}_{H,\mathcal{U}}$, and thus $b' = \varphi(b)$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$. In other words, every element in $\mathcal{D}_{H,\mathcal{P}}$ with less H-complexity than a' is invertible.

Hence, Proposition 4.3.13 applies and tells us that $a' = f_\omega(\bar{a})$ for some $\bar{a} = \sum_k a_k \in$ $\mathcal{D}_{N,\mathcal{U}_N}((t;\tilde{\tau}))$, where $a_k \in \mathcal{D}_{N,\mathcal{U}_N} t^k$ and $\mathrm{Tree}_H(f_\omega(a_k)) \leq \mathrm{Tree}_H(a')$. Moreover, we claim that there are at least two non-zero summands. Otherwise, if $\bar{a} = a_n$, then $a' = f_\omega(\bar{a}) =$ $f_\omega(a_n)$ and in particular $\mathrm{Tree}_H(a') = \mathrm{Tree}_H(f_\omega(a_n))$. Thus, the same proposition tells us that $\alpha \in \mathrm{Rat}(K^\times N)x^n \subseteq \mathrm{Rat}(K^\times N)K^\times H$, and then Theorem 4.3.8(iv) states that source$(\alpha) \leq K^\times N$, what implies that $H \leq N$, a contradiction.

Hence, $\mathrm{Tree}_H(f_\omega(a_k)) < \mathrm{Tree}_H(a')$ for all k. In particular this is true for n, the smallest k such that a_k is non-zero. Assume that $a_n = yt^n$ for some $y \in \mathcal{D}_{N,\mathcal{U}_N}$. Remark 4.3.12(2.) tells us that f_ω defines an isomorphism between $\mathcal{D}_{N,\mathcal{U}_N}$ and $\mathcal{D}_{N,\mathcal{P}}$, and therefore $f_\omega(y) \in \mathcal{D}_{N,\mathcal{P}} \subseteq \mathcal{D}_{H,\mathcal{P}}$. Since $t^n = \tilde{\phi}(x^n)$, we also have $f_\omega(t^n) = f_\omega\tilde{\phi}(x^n) \in f_\omega\tilde{\phi}(K[H]) \subseteq \mathcal{D}_{H,\mathcal{P}}$. As a consequence $f_\omega(a_n)$ is an element in $\mathcal{D}_{H,\mathcal{P}}$ with strictly less H-complexity than a', so as before we have that $f_\omega(a_n)$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$. Since t is invertible, this implies that $f_\omega(y)$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$, and hence in $\mathcal{D}_{N,\mathcal{P}}$. Again, since f_ω is an isomorphism from $\mathcal{D}_{N,\mathcal{U}_N}$ to $\mathcal{D}_{N,\mathcal{P}}$, we conclude that y is invertible in $\mathcal{D}_{N,\mathcal{U}_N}$. This implies that $\bar{a}$ is invertible in $\mathcal{D}_{N,\mathcal{U}_N}((t;\tilde{\tau}))$, and therefore $f_\omega(\bar{a}) = a'$ is invertible in $\mathcal{P}$, and hence in $\mathcal{D}_{H,\mathcal{P}}$ since it is division closed. Finally, since φ is an isomorphism from $\mathcal{D}_{H,\mathcal{U}}$ to $\mathcal{D}_{H,\mathcal{P}}$ and $a = \varphi^{-1}(a')$, we conclude that a is invertible in $\mathcal{D}_{H,\mathcal{U}}$, and hence in $\mathcal{D}$, as we wanted to show.

Thus, we have just proved that $\mathcal{D}$ is a division ring. Since $\phi : K[G] \to \mathcal{U}$ is epic and factors as $\phi : K[G] \to \mathcal{D} \hookrightarrow \mathcal{U}$, we see that $\mathcal{D} \hookrightarrow \mathcal{U}$ is epic. Since $\mathcal{D}$ is a division ring this implies that $\mathcal{U} = \mathcal{D}$ (see Proposition 4.1.14). □

The most important consequence of this is the aforementioned strong Atiyah conjecture for locally indicable groups.

Theorem 4.4.2. *Let G be a locally indicable group. Then G satisfies the strong Atiyah conjecture over $\mathbb{C}$ and $\mathcal{R}_{\mathbb{C}[G]} = \mathcal{D}_{\mathbb{C}[G]}$ is the Hughes-free division $\mathbb{C}[G]$-ring of fractions.*

Proof. By Proposition 4.2.1(1.), $\mathcal{U}(G)$ is a positive definite $*$-regular ring containing $\mathbb{C}[G]$ as a $*$-subring. Furthermore, $\mathcal{U}(G)$ comes equipped with a faithful Sylvester matrix rank function rk_G (Proposition 4.2.2), and hence by definition its restriction to $\mathbb{C}[G]$ is a $*$-regular rank function on $\mathbb{C}[G]$, which is in addition Hughes-free by Proposition 4.2.7. Its $*$-regular envelope is precisely $(\mathcal{R}_{\mathbb{C}[G]}, \mathrm{rk}_G, \iota)$, where ι denotes the inclusion map, and $\mathcal{R}_{\mathbb{C}[G]}$ is positive definite because $\mathcal{U}(G)$ is positive definite. Thus, we can apply Theorem 4.4.1 to conclude that $\mathcal{R}_{\mathbb{C}[G]}$ is a division ring and coincides with $\mathcal{D}_{\mathbb{C}[G]}$, the division closure of $\mathbb{C}[G]$ in $\mathcal{U}(G)$. Therefore G satisfies the strong Atiyah conjecture over $\mathbb{C}$ by Proposition 4.2.5(5.). The last assertion follows then from Proposition 3.4.31. □

Corollary 4.4.3. *Let G be a locally indicable group and K a subfield of $\mathbb{C}$. Then there exists a Hughes-free division $K[G]$-ring of fractions. In particular, if G is countable, then G satisfies the strong Atiyah conjecture over K and the division closure $\mathcal{D}_{K[G]}$ of $K[G]$ in $\mathcal{U}(G)$ is the Hughes-free division $K[G]$-ring of fractions.*

Proof. Let first G be countable. By Theorem 4.4.2, $\mathcal{D}_{\mathbb{C}} := \mathcal{D}_{\mathbb{C}[G]} = \mathcal{R}_{\mathbb{C}[G]}$ is the Hughes-free division $\mathbb{C}[G]$-ring of fractions. Since $\mathcal{D}_K := \mathcal{D}_{K[G]} \subseteq \mathcal{D}_{\mathbb{C}}$ is division closed in $\mathcal{U}(G)$, it is also a division ring, and G satisfies the strong Atiyah conjecture over K by Proposition 4.2.5(5).

Let H be a finitely generated subgroup of G, $N \lhd H$ with H/N infinite cyclic, and $x \in H$ such that $H/N = \langle Nx \rangle$. Consider the division closures $\mathcal{D}_{K[N],\mathcal{D}_K}$ $\mathcal{D}_{K[N],\mathcal{D}_{\mathbb{C}}}$, $\mathcal{D}_{\mathbb{C}[N],\mathcal{D}_{\mathbb{C}}}$. Since $\mathcal{D}_K \subseteq \mathcal{D}_{\mathbb{C}}$ are division rings and $K[N] \subseteq \mathbb{C}[N]$, we have $\mathcal{D}_{K[N],\mathcal{D}_K} =$

$\mathcal{D}_{K[N],\mathcal{D}_{\mathbb{C}}} \subseteq \mathcal{D}_{\mathbb{C}[N],\mathcal{D}_{\mathbb{C}}}$. Now, the Hughes-freeness of $\mathcal{D}_G$ implies that the powers of x are $\mathcal{D}_{\mathbb{C}[N],\mathcal{D}_{\mathbb{C}}}$-linearly independent and therefore $\mathcal{D}_{K[N],\mathcal{D}_K}$-linearly independent. In view of Lemma 3.4.22 this implies that $\mathcal{D}_K$ is the Hughes-free division $K[G]$-ring of fractions.

Now, let G be arbitrary. The previous reasoning shows that every finitely generated subgroup G' of G admits a Hughes-free division $K[G']$-ring of fractions, and hence the same holds for G (see [Sán08, Corollary 6.6(i)]). The Hughes-free division $K[G]$-ring of fractions can be built as the direct limit $\varinjlim_{G' \leq_{f.g.} G} \mathcal{D}_{K[G']}$. □

Remark. Although we decided to use a general argument in the proof, we showed in Section 4.2 that the diagram

$$\begin{array}{ccccc} K[N] & \longrightarrow & K[H] & \longrightarrow & K[G] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{U}(N) & \longrightarrow & \mathcal{U}(H) & \longrightarrow & \mathcal{U}(G). \end{array}$$

is commutative, and we mentioned that when $N \lhd H$, the powers of x (with $H/N = \langle Nx \rangle$ infinite cyclic) are already left $\mathcal{U}(N)$-linearly independent (cf. [Lin98, Lemmas 9.2 & 9.3]). Since one can show by regularity of $\mathcal{U}(N)$ that

$$\mathcal{D}_{K[N],\mathcal{D}_K} = \mathcal{D}_{K[N],\mathcal{U}(G)} = \mathcal{D}_{K[N],\mathcal{U}(N)} \subseteq \mathcal{U}(N),$$

this implies that the powers of x are left $\mathcal{D}_{K[N],\mathcal{D}_K}$-linearly independent. □

We need to point out that it was already known that every non-zero element of $\mathbb{C}[G]$ was invertible in $\mathcal{U}(G)$, even for left orderable groups ([Lin92, Theorem 2], or [DL07, Theorem 3.3]). In the latter paper it is also proved that in every left orderable group G with homological dimension (with respect to $\mathbb{Z}$) at most 1, every two-generator subgroup is free. In [KLL09, Theorem 2] it is stated that a group of homological dimension at most 1 satisfying the Atiyah conjecture is *locally free*, i.e., every finitely generated subgroup is free. Thus, as a consequence of Theorem 4.4.2, we have the following.

Corollary 4.4.4. *Any locally indicable group of homological dimension at most one is locally free.*

By mixing Corollary 4.4.3, with Theorem 3.5.13 and Proposition 3.4.26 we also obtain the following result.

Corollary 4.4.5. *Let K be a subfield of $\mathbb{C}$, G a countable group arising as an extension*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is a free group. Then $K[G]$ is a pseudo-Sylvester domain and $\mathcal{D}_{K[G]}$, the division closure of $K[G]$ in $\mathcal{U}(G)$, is the universal $K[G]$-ring of fractions.

The fact that G satisfies the strong Atiyah conjecture and $\mathcal{D}_{K[G]}$ is Hughes-free for the family of groups considered in the corollary was already known, and the first proof goes back to Linnell ([Lin93], see also [Lüc02, Chapter 10]), since they all lie in Linnell's class $\mathcal{C}$.

In Linnell's proof, Hughes-freeness was already used to identify $\mathcal{D}_{\mathbb{C}[F]}$ with the universal field of fractions of $\mathbb{C}[F]$ as $\mathbb{C}[F]$-rings, and the same arguments apply for any subfield K of $\mathbb{C}$. Using this, one can directly exhibit $\mathcal{D}_{K[G]}$ as the Ore division ring of $\mathcal{D}_{K[F]} * \mathbb{Z}$. Indeed, we have seen in Section 4.2 that this crossed product can be built as a subring of $\mathcal{U}(G)$ (cf. [Lüc02, Lemma 10.57(1)]), and hence, inasmuch as $\mathcal{D}_{K[G]}$ is a division ring containing $\mathcal{D}_{K[F]} * \mathbb{Z}$, the universal property of the Ore localization tells us that it also contains the ring $\mathcal{Q}(\mathcal{D}_{K[F]} * \mathbb{Z})$. Since the latter is a division subring containing $K[G]$, necessarily $\mathcal{D}_{K[G]} = \mathcal{Q}(\mathcal{D}_{K[F]} * \mathbb{Z})$.

We finish the section with another result regarding the stability of the strong Atiyah conjecture under extensions by locally indicable groups. This result was pointed out to us by Fabian Henneke and Dawid Kielak.

Proposition 4.4.6. *Let K be a subfield of $\mathbb{C}$, G_2 a countable group arising as an extension*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

where G_1 is a torsion-free normal subgroup of G_2 and G_3 is locally indicable. If G_1 satisfies the strong Atiyah conjecture over K, then G_2 satisfies the strong Atiyah conjecture over K.

Proof. First of all, note that in the previous conditions G_2 is torsion-free, since an element $g \in G_2$ either lies in G_1, which is torsion-free, or maps to a non-zero element in G_3, which is also torsion-free, from where g must have infinite order. Thus, we are going to show that G satisfies Proposition 4.2.5(5). Let $\mathcal{D}_{K[G_2]}$ be the division closure of $K[G_2]$ in $\mathcal{U}(G_2)$, and consider the universal morphism of rational $K^\times G_2$-semirings $\Phi : \mathrm{Rat}(K^\times G_2) \to \mathcal{D}_{K[G_2]}$. By induction on the G_2-complexity Tree_{G_2} we will show that any non-zero element $a \in \mathcal{D}$ is invertible.

If $\mathrm{Tree}_{G_2}(a) = 1_{\mathcal{T}}$ and α realizes the G_2-complexity of a, then $\alpha \in K^\times G_2$ by Lemma 4.3.7(ii), and therefore $a = \Phi(\alpha) = \alpha \in K^\times G_2$ is invertible.

Now assume that $\mathrm{Tree}_G(a) > 1_{\mathcal{T}}$ and that for every $0 \neq b \in \mathcal{D}_{K[G_2]}$ such that $\mathrm{Tree}_{G_2}(b) < \mathrm{Tree}_{G_2}(a)$, b is invertible. Let $\alpha \in \mathrm{Rat}(K^\times G_2)$ realize the G_2-complexity of a. As in the proof of Theorem 4.3.14, we can assume without loss of generality that α is primitive, so that $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha))$. Consider the map $\pi_{G_2} : K^\times G_2 \to K^\times G_2/K^\times \cong G_2$ and set $H = \pi_{G_2}(\mathrm{source}(\alpha))$, which is a finitely generated subgroup of G_2. By construction we have $\mathrm{source}(\alpha) \leq K^\times H$, and hence by Proposition 4.3.9 we deduce that $a = \Phi(\alpha) \in \mathcal{D}_{H,\mathcal{U}(G_2)}$, the division closure of $K[H]$ in $\mathcal{U}(G_2)$. Another consequence is that $\alpha \in \mathrm{Rat}(K^\times H)$, and hence

$$\mathrm{Tree}_H(a) = \mathrm{Tree}_G(a) = \mathrm{Tree}(\alpha).$$

If $H \leq G_1$, then $K[H] \subseteq K[G_1] \subseteq \mathcal{U}(G_1)$, and hence by regularity of $\mathcal{U}(G_1)$ and commutativity of

$$\begin{tikzcd} K[G_1] \arrow[r] \arrow[d] & K[G_2] \arrow[d] \\ \mathcal{U}(G_1) \arrow[r] & \mathcal{U}(G_2) \end{tikzcd}$$

we deduce that (see Lemma 3.3.3)

$$a \in \mathcal{D}_{H,\mathcal{U}(G_2)} = \mathcal{D}_{H,\mathcal{U}(G_1)} \subseteq \mathcal{D}_{G_1,\mathcal{U}(G_1)} = \mathcal{D}_{K[G_1]}.$$

By hypothesis, G_1 is torsion-free and satisfies the Atiyah conjecture over K, and hence $\mathcal{D}_{K[G_1]}$ is a division ring. Therefore, a is invertible in $\mathcal{U}(G_1)$, hence in $\mathcal{D}_{H,\mathcal{U}(G_1)} = \mathcal{D}_{H,\mathcal{U}(G_2)}$, and consequently in $\mathcal{D}_{G_2,\mathcal{U}(G_2)} = \mathcal{D}_{K[G_2]}$.

Otherwise, the image $p(H)$ of H under $G_2 \xrightarrow{p} G_3$ is a non-trivial finitely generated subgroup of G_3, and hence $p(H)$ is indicable. Thus, H is also indicable, and hence there exists a normal subgroup $N \lhd H$ such that H/N is infinite cyclic. Moreover, by Proposition 4.2.7, we obtain that rk_H, as a Sylvester matrix rank function on $\mathbb{C}[H]$, is the natural extension of rk_N as a Sylvester matrix rank function on $\mathbb{C}[N]$. Since $(\mathcal{R}_{\mathbb{C}[G_2]}, \mathrm{rk}_{G_2}, \iota)$, where ι denotes inclusion, is the positive definite $*$-regular envelope of the $*$-regular rank rk_{G_2} on $\mathbb{C}[G_2]$, we can still proceed as in Section 4.1.1 for this particular choice of N and H to get the diagram Eq. (4.4) over $\mathbb{C}$

$$\begin{tikzcd} \mathbb{C}[N] \arrow[r, hook] \arrow[d, hook] & \mathbb{C}[H] \arrow[r, hook] \arrow[d, "\tilde{j}"'] & \mathcal{U}_H \arrow[d, "\varphi"] \\ \mathcal{U}_N \arrow[r, hook] & \mathcal{U}_N((t;\tilde{\tau})) \arrow[r, "f_\omega"'] & \mathcal{P}, \end{tikzcd}$$

where $\mathcal{U}_N$ and $\mathcal{U}_H$ denote, respectively, the $*$-regular closures of $\mathbb{C}[N]$ and $\mathbb{C}[H]$ in $\mathcal{R}_{\mathbb{C}[G_2]}$, f_ω and φ are injective, and $\tilde{j}$ is the map acting as the identity on $\mathbb{C}[N]$ and sending $x \mapsto t$. If we restrict the first two maps to $K[N]$ and $K[H]$, we still have a commutative diagram

$$\begin{tikzcd} K[N] \arrow[r, hook] \arrow[d, hook] & K[H] \arrow[r, hook] \arrow[d, "\tilde{j}"'] & \mathcal{U}_H \arrow[d, "\varphi"] \\ \mathcal{U}_N \arrow[r, hook] & \mathcal{U}_N((t;\tilde{\tau})) \arrow[r, "f_\omega"'] & \mathcal{P}. \end{tikzcd}$$

Thus, the analog of observations a.) and b.) at the beginning of the section apply here, i.e.,

a'.) First, $\mathcal{A} = \mathcal{U}_N$ together with the inclusion map is a regular $K[N]$-ring. Second, recall that, if $x \in H$ is such that $H/N = \langle Nx \rangle$ is infinite cyclic and τ denotes the automorphism of $\mathbb{C}[N]$ given by left conjugation by x, then $\tilde{\tau}$ is an automorphism of $\mathcal{A}$ satisfying $\tilde{\tau} \circ j = j \circ \tau$ (where j denotes just the inclusion map $\mathbb{C}[N] \hookrightarrow \mathcal{U}_N$).

Since τ restricts to the corresponding automorphism of $K[N]$, it is still true that this commutes

$$\begin{tikzcd} K[N] \arrow[r, "\tau"] \arrow[d, "j"'] & K[N] \arrow[d, "j"] \\ \mathcal{A} \arrow[r, "\tilde{\tau}"'] & \mathcal{A}. \end{tikzcd}$$

Thus, $\mathcal{A}$, $\tilde{\tau}$ and $(\mathcal{P}, f_\omega)$ satisfy the conditions (i), (ii) and (iii) needed in order to apply Proposition 4.3.13, and $\tilde{j}$ is the actual $\tilde{\phi}$ constructed in Remark 4.3.12(1.).

b'.) Let $\mathcal{D}_{H,\mathcal{R}_{\mathbb{C}[G_2]}}$ and $\mathcal{D}_{H,\mathcal{P}}$ denote, respectively, the division closure of $K[H]$ and $f_\omega\tilde{j}(K[H])$ in $\mathcal{R}_{\mathbb{C}[G_2]}$ and $\mathcal{P}$. Note that since $\mathcal{R}_{\mathbb{C}[G_2]}$ is regular, $\mathcal{D}_{H,\mathcal{R}_{\mathbb{C}[G_2]}} = \mathcal{D}_{H,\mathcal{U}(G_2)}$. Moreover, since $\mathcal{U}_H$ is regular and φ is an embedding, $\varphi(\mathcal{U}_H)$ is regular and by Lemma 3.3.3(2) and (3) we have that φ restricts to an isomorphism of $K[H]$-rings from $\mathcal{D}_{H,\mathcal{U}_H}$ to $\mathcal{D}_{H,\varphi(\mathcal{U}_H)}$ and that

$$\mathcal{D}_{H,\mathcal{U}(G_2)} = \mathcal{D}_{H,\mathcal{R}_{\mathbb{C}[G_2]}} = \mathcal{D}_{H,\mathcal{U}_H} \overset{\varphi}{\cong} \mathcal{D}_{H,\varphi(\mathcal{U}_H)} = \mathcal{D}_{H,\mathcal{P}}.$$

Thus, the same proof of Theorem 4.4.1 from this point on shows that a must be invertible over $\mathcal{D}_{H,\mathcal{U}(G_2)}$, and hence in $\mathcal{D}_{K[G_2]}$. Therefore, $\mathcal{D}_{K[G_2]}$ is a division ring and G_2 satisfies the strong Atiyah conjecture over K. □

Chapter 5

Related conjectures and results

This chapter, which is based on [JL20, Sections 6 to 8], is devoted to the study of results and conjectures that are related to the strong Atiyah conjecture, either in the sense that they are obtained as a consequence of the fact that locally indicable groups G satisfy the conjecture (and have a Hughes-free $K[G]$-ring of fractions), or in the sense that the same methods developed for its proof apply in different contexts.

The chapter is divided in three sections. In Section 5.1, we use the existence of the Hughes-free division ring of fractions to prove other related conjectures posed by A. Jaikin in [Jai19], together with some other corollaries that follow from them. In Section 5.2 we introduce Lück's approximation conjecture in the space of marked groups. We start by proving some results regarding the comparison between different rank functions and we use them to prove that the conjecture holds whenever the group being approximated is virtually locally indicable. Finally, in Section 5.3 we study further the question of whether the Hughes-free division ring of fractions is also universal.

5.1 Other directly related conjectures

In this subsection, we make use of the existence and uniqueness of the Hughes-free division ring of fractions to prove some other results regarding the group ring $K[G]$ where G is locally indicable. The propositions in this subsection, which have been given a name, correspond to questions that were solved for sofic groups in [Jai19, Corollaries 1.5, 1.6 & 1.7]. The statements here, in terms of division closures, are equivalent to the original ones when the fields considered are subfields of $\mathbb{C}$ closed under complex conjugation, because we already know that these objects are division rings.

Proposition 5.1.1 (The independence conjecture). *Let G be a countable locally indicable group, K a field of characteristic zero and $\varphi_1, \varphi_2 : K \to \mathbb{C}$ two different embeddings of K into $\mathbb{C}$. Then, for every matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$,*

$$\mathrm{rk}_G(\varphi_1(A)) = \mathrm{rk}_G(\varphi_2(A)).$$

Proof. Let us denote $\varphi_1(K) = K_1$ and $\varphi_2(K) = K_2$. These homomorphisms extend to isomorphisms $\varphi_i : K[G] \to K_i[G]$ by setting $\varphi_i(ag) = \varphi_i(a)g$. Now, Corollary 4.4.3 tells us that the division closure $\mathcal{D}_{K_i[G]}$ of $K_i[G]$ in $\mathcal{U}(G)$ is the Hughes-free division $K_i[G]$-ring of fractions, $i = 1, 2$. Therefore, $\mathcal{D}_{K_i[G]}$ is a Hughes-free division $K[G]$-ring of fractions for $i = 1, 2$, and hence by uniqueness of the Hughes-free division ring (Theorem 3.4.23) there exists an isomorphism $\psi : \mathcal{D}_{K_1[G]} \to \mathcal{D}_{K_2[G]}$ such that the following commutes

$$\begin{tikzcd} & K[G] \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \mathcal{D}_{K_1[G]} \arrow[rr, "\psi"'] & & \mathcal{D}_{K_2[G]} \end{tikzcd}$$

Since in a division ring there exists only one Sylvester matrix rank function, it must be the case that $\psi^\sharp(\mathrm{rk}_G) = \mathrm{rk}_G$. Thus, if $A \in \mathrm{Mat}(K[G])$, we have

$$\mathrm{rk}_G(\varphi_1(A)) = \mathrm{rk}_G(\psi(\varphi_1(A))) = \mathrm{rk}_G(\varphi_2(A)).$$

□

An important consequence of the independence conjecture is the following.

Corollary 5.1.2. *Let G be a locally indicable group and K a field of characteristic zero. Then there exists a Hughes-free division $K[G]$-ring of fractions.*

Proof. In view of [Sán08, Corollary 6.6](i), the result holds for an arbitrary locally indicable G if it holds for every finitely generated subgroup of G. Hence, we can assume that G is finitely generated.

Let A be a matrix over $K[G]$. We can find a finitely generated (over the prime field $\mathbb{Q}$) subfield K_1 of K such that $A \in \mathrm{Mat}(K_1[G])$, and hence an embedding $\varphi_1 : K_1 \to \mathbb{C}$, which extends to $\varphi_1 : K_1[G] \to \mathbb{C}[G]$. Set

$$\mathrm{rk}_{K_1}(A) = \mathrm{rk}_G(\varphi_1(A)).$$

By Proposition 5.1.1, the value $\mathrm{rk}_{K_1}(A)$ does not depend on the choice of φ_1. In addition, observe that rk_{K_1} is a Sylvester matrix rank function on $K_1[G]$.

Moreover, if K_2 is another finitely generated subfield of K such that $A \in \mathrm{Mat}(K_2[G])$, then $\mathrm{rk}_{K_1}(A) = \mathrm{rk}_{K_2}(A)$. Indeed, in this case we can consider the subfield K_0 generated by K_1 and K_2, so that $A \in \mathrm{Mat}(K_0[G])$. If $\varphi_0 : K_0 \to \mathbb{C}$ is an embedding, then it restricts to embeddings of K_1 and K_2 into $\mathbb{C}$, and hence since the values $\mathrm{rk}_{K_i}(A)$ do not depend on the embedding,

$$\mathrm{rk}_{K_2}(A) = \mathrm{rk}_G(\varphi_0(A)) = \mathrm{rk}_{K_1}(A).$$

Therefore, the value $\mathrm{rk}(A) = \mathrm{rk}_L(A)$ if L is a finitely generated subfield of K and $A \in \mathrm{Mat}(L[G])$ is well-defined. Moreover, rk defines a Sylvester matrix rank function on $K[G]$ because for any matrices A, B, C over $K[G]$, we can find a finitely generated subfield L of K such that $A, B, C \in \mathrm{Mat}(L[G])$ and rk_L is a Sylvester matrix rank function on $L[G]$.

Thus, we have constructed a Sylvester matrix rank function rk on $K[G]$ which takes only integer values (because rk_G does on matrices over $\mathbb{C}[G]$). Therefore, by Corollary 3.1.17 it has an epic division envelope $(\mathcal{D}, \phi)$, i.e., a division ring $\mathcal{D}$ together with a ring homomorphism $\phi : K[G] \to \mathcal{D}$ such that $\mathrm{rk} = \phi^\sharp(\mathrm{rk}_\mathcal{D})$. The map ϕ is furthermore injective because rk_G is faithful and the maps φ_i are injective.

It is left to show that $\mathcal{D}$ is the Hughes-free division $K[G]$-ring of fractions. Let H be a finitely generated subgroup of G, $N \lhd H$ such that H/N is infinite cyclic and $x \in H$ such that $H/N = \langle Nx \rangle$. If $\mathcal{D}_{K[N],\mathcal{D}}$ denotes the division closure of $\phi(K[N])$ in $\mathcal{D}$, we need to show that the powers of $\phi(x)$ are left $\mathcal{D}_{K[N],\mathcal{D}}$-linearly independent, so consider any expression of the form

$$a_0 + a_1\phi(x) + \cdots + a_n\phi(x)^n = 0$$

with $a_i \in \mathcal{D}_{K[N],\mathcal{D}}$.

Observe that for every two finitely generated subfields K_1 and K_2 of K, there exists a finitely generated subfield K_3 containing both, and hence for every subgroup G' of G we have that $\mathcal{D}_{K_1[G'],\mathcal{D}}$ and $\mathcal{D}_{K_2[G'],\mathcal{D}}$ are contained in $\mathcal{D}_{K_3[G'],\mathcal{D}}$. This shows that $\mathcal{S} = \bigcup_{L \subseteq_{f.g.} K} \mathcal{D}_{L[G'],\mathcal{D}}$ is a ring and one can show using the inductive construction of $\mathcal{D}_{K[G'],\mathcal{D}}$ that $\mathcal{D}_{K[G'],\mathcal{D}} = \mathcal{S}$. Therefore, we can find a finitely generated subfield K_0 of K such that $a_i \in \mathcal{D}_{K_0[N],\mathcal{D}}$ for $i = 0, \ldots, n$. Let $\varphi : K_0 \to \mathbb{C}$ be an embedding with image $\varphi(K_0) = K_0'$ and observe that we have the following

$$\begin{tikzcd} & K_0[G] \arrow[dl, "\phi"'] \arrow[dr, "\varphi"] & \\ \mathcal{D}_{K_0[G],\mathcal{D}} \arrow[rr, dashed, "\psi"'] & & \mathcal{D}_{K_0'[G]} \end{tikzcd}$$

where, as usual, $\mathcal{D}_{K_0'[G]}$ denotes the division closure of $K_0'[G]$ in $\mathcal{U}(G)$. By definition we had $\mathrm{rk}_{K_0} = \varphi^\sharp(\mathrm{rk}_G)$, and by definition of $\mathcal{D}$ we also have $\mathrm{rk}_{K_0} = \phi^\sharp(\mathrm{rk}_\mathcal{D})$. By uniqueness of the epic division envelope (Corollary 3.1.17) there exists an isomorphism ψ as indicated in the diagram. But, as in the proof of Proposition 5.1.1, $\mathcal{D}_{K_0'[G]}$ is the Hughes-free division $K_0[G]$-ring of fractions. Since $\phi(K_0[N]) \subseteq \mathcal{D}_{K_0[G],\mathcal{D}}$, which is a division ring, $\mathcal{D}_{K_0[N],\mathcal{D}}$ equals the division closure of $\phi(K_0[N])$ in $\mathcal{D}_{K_0[G],\mathcal{D}}$, and hence Lemma 3.3.3(3) tells us that ψ restricts to an isomorphism of $K_0[N]$-rings $\mathcal{D}_{K_0[N],\mathcal{D}} \to \mathcal{D}_{K_0'[N]}$. In particular, the previous expression goes through ψ to an expression in the powers of $\varphi(x)$ with coefficients in $\mathcal{D}_{K_0'[N]}$, and hence by Hughes-freeneess and injectivity of ψ, we get $a_0 = \cdots = a_n = 0$, as we wanted to show. $\square$

For a field K of non-zero characteristic and a locally indicable group G it is still unknown (at the time of writing) whether there exists a Hughes-free division $K[G]$-ring of fractions $\mathcal{D}$. However, if such $\mathcal{D}$ exists and we are given a field extension L/K we can construct, under some extra assumptions, the Hughes-free division $L[G]$-ring of fractions from $\mathcal{D}$.

Since we shall be dealing with tensor products of the form $\mathcal{D} \otimes_K L$ where $\mathcal{D}$ is the Hughes-free division $K[G]$-ring of fractions, let us record some generalities.

Remark 5.1.3.

1. Recall from Corollary 4.1.12 that since $K[G] \hookrightarrow \mathcal{D}$ is epic, we have $K \subseteq Z(\mathcal{D})$, what makes $\mathcal{D}$, and hence $\mathcal{D} \otimes_K L$, a K-algebra (cf. [Rot09, Proposition 2.60]). In addition, the map $L[G] \to K[G] \otimes_K L$ given by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G}(g \otimes a_g)$ is a K-algebra isomorphism, and hence, since $\Box \otimes_K L$ is exact (because K is a field), $L[G]$ embeds in $\mathcal{D} \otimes_K L$ via $L[G] \cong K[G] \otimes_K L \hookrightarrow \mathcal{D} \otimes_K L$. Similarly, since $\mathcal{D} \otimes_K L$ is non-zero, the map $\mathcal{D} \to \mathcal{D} \otimes_K L$ given by $d \mapsto d \otimes 1$ is an embedding, and the following commutes
$$\begin{tikzcd} K[G] \arrow[r] \arrow[d] & L[G] \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{D} \otimes_K L \end{tikzcd} \tag{5.1}$$

2. Since $K \subseteq Z(\mathcal{D})$, we also have $K[t] \subseteq Z(\mathcal{D}[t])$. This implies that the group homomorphism $\mathcal{D} \otimes_K K[t] \to \mathcal{D}[t]$ induced by the K-biadditive map $\mathcal{D} \times K[t] \to \mathcal{D}[t]$ with $(d, p) \mapsto dp$ is actually a ring homomorphism and, in fact, a ring isomorphism $\mathcal{D} \otimes_K K[t] \cong \mathcal{D}[t]$ acting on generators as $d \otimes p \mapsto dp$. Similarly, since $\mathcal{D}[t]$ is a domain, $T = K[t]\setminus\{0\}$ is a multiplicative subset of non-zero-divisors in $K[t]$ and $\mathcal{D}[t]$ satisfying both Ore conditions (since T lies in the center) and we have an isomorphism $\mathcal{D} \otimes_K K(t) = \mathcal{D} \otimes_K K[t]T^{-1} \cong \mathcal{D}[t]T^{-1}$ given by $d \otimes \frac{p}{q} \mapsto (dp)q^{-1}$. Since $\mathcal{D}[t]$ is an Ore domain by Example 3.1.7 the universal property of Ore localization Proposition 3.1.4 gives us a commutative diagram
$$\begin{tikzcd} \mathcal{D} \otimes_K K[t] \arrow[r, "\cong"] \arrow[d] & \mathcal{D}[t] \arrow[d] \arrow[dr] & \\ \mathcal{D} \otimes_K K(t) \arrow[r, "\cong"] & \mathcal{D}[t]T^{-1} \arrow[r] & \mathcal{D}(t) \end{tikzcd}$$
Note that $\mathcal{D}[t]T^{-1}$ is also an Ore domain (cf. [GW04, Exercise 6C & Corollary 6.7]) with Ore division ring $\mathcal{Q}(\mathcal{D}[t]T^{-1}) = \mathcal{D}(t)$. Thus, the same holds for $\mathcal{D} \otimes_K K(t)$ and we obtain a commutative diagram
$$\begin{tikzcd} \mathcal{D} \otimes_K K(t) \arrow[r, "\cong"] \arrow[d] & \mathcal{D}[t]T^{-1} \arrow[d] \\ \mathcal{Q}(\mathcal{D} \otimes_K K(t)) \arrow[r, "\cong"] & \mathcal{D}(t) \end{tikzcd} \tag{5.2}$$
For every $d \in \mathcal{D}$ observe that $d \otimes 1 \mapsto d$.

3. More generally, if L is an extension of K such that $\mathcal{D} \otimes_K L$ is a domain, then it is an Ore domain. Indeed, first note that for any subfield L' of L which is a finitely generated extension of K, the tensor product $\mathcal{D} \otimes_K L'$ is noetherian by a version

of the Hilbert basis theorem. More precisely, since L'/K is finitely generated, $\mathcal{D} \otimes_K L'$ is a localization of a quotient of a polynomial ring (in a finite number of indeterminates) with coefficients in $\mathcal{D}$, and each of these operations preserves the noetherianity of $\mathcal{D}$ (see [GW04, Proposition 1.2, Theorem 1.9 & Corollary 10.16]). Since $\mathcal{D} \otimes_K L'$ is a subring of $\mathcal{D} \otimes_K L$ ($\mathcal{D} \otimes_K \square$ is exact because K is a field) it is also a domain and hence an Ore domain by [GW04, Corollary 6.7]. Consequently, since every pair of elements of $\mathcal{D} \otimes_K L$ live in an appropriate $\mathcal{D} \otimes_K L'$, $\mathcal{D} \otimes_K L$ is also an Ore domain.

4. If $Z(\mathcal{D}) = K$, then $\mathcal{D} \otimes_K L$ is a simple ring (see [Pie82, §12.4 Lemma b (ii)]).

$\square$

Lemma 5.1.4. *Let G be a locally indicable group, K a field and L/K a field extension. If there exists a Hughes-free division $K[G]$-ring of fractions $\mathcal{D}$ and $\mathcal{D} \otimes_K L$ is a domain, then the Ore division ring $\mathcal{Q}(\mathcal{D} \otimes_K L)$ is a Hughes-free division $L[G]$-ring of fractions.*

Proof. In view of Remark 5.1.3(3.), it makes sense to consider the Ore division ring $\mathcal{Q}(\mathcal{D} \otimes_K L)$ of $\mathcal{D} \otimes_K L$. For any subgroup $N \leq G$, let $\mathcal{D}_{N,\mathcal{D}}$ denote the division closure of $K[N]$ in $\mathcal{D}$. Identifying $L[G] \cong K[G] \otimes_K L$, we have that the division closure $\mathcal{S}$ of $L[N]$ in $\mathcal{Q}(\mathcal{D} \otimes_K L)$ is $\mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)$. Indeed, as before, it makes sense to consider $\mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)$, a division subring of $\mathcal{Q}(\mathcal{D} \otimes_K L)$, and since $L[N] \cong K[N] \otimes_K L \subseteq \mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)$, we conclude that $\mathcal{S} \subseteq \mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)$. Conversely, since $\mathcal{D}_{N,\mathcal{D}}$ is generated by $K[N]$ as a division ring, one can inductively see that every generator (and hence every element) of $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$ lives in the division ring $\mathcal{S}$. By the universal property of Ore localization we obtain that $\mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L) \subseteq \mathcal{S}$, so that $\mathcal{S} = \mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)$.

Therefore, proving that $\mathcal{Q}(\mathcal{D} \otimes_K L)$ is Hughes-free amounts to see that for every finitely generated subgroup $H \leq G$, every $N \triangleleft H$ and $x \in H$ such that $H/N = \langle Nx \rangle$ is infinite cyclic, the powers of $x \otimes 1$ are (left) $\mathcal{Q}(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)$-linearly independent. Clearing denominators, it suffices to prove $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$-linear independence. To see this, let R denote the subring of $\mathcal{D}$ generated by $\mathcal{D}_{N,\mathcal{D}}$ and x, and S denote the subring of $\mathcal{D} \otimes_K L$ generated by $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$ and $x \otimes 1$. We claim that

$$S \overset{1.}{=} R \otimes_K L \overset{2.}{\cong} \mathcal{D}_{N,\mathcal{D}}[t;\tilde{\tau}] \otimes_K L \overset{3.}{\cong} (\mathcal{D}_{N,\mathcal{D}} \otimes_K L)[t;\tilde{\tau} \otimes \mathrm{id}_L]$$

where $\tilde{\tau}$ denotes the automorphism of $\mathcal{D}_{N,\mathcal{D}}$ induced by left conjugation by x. Indeed, this is because of the following,

1. On the one hand, $R \otimes_K L$ contains $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$ and $x \otimes 1$, so $S \subseteq R \otimes_K L$. On the other hand, every generator of $R \otimes_K L$ can be expressed using sums, substractions and products of elements in $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$ and powers of $x \otimes 1$, so that we have the other containment, and hence equality.

2. This follows from the Hughes-freeness of $\mathcal{D}$ (the powers of x are $\mathcal{D}_{N,\mathcal{D}}$-linearly independent). This isomorphism sends $x \otimes 1$ to $t \otimes 1$.

3. Since $K \subseteq Z(\mathcal{D})$ (Corollary 4.1.12), $\tilde{\tau}$ leaves the elements of K fixed, and hence it is an automorphism of $\mathcal{D}_{N,\mathcal{D}}$ as a K-algebra. Using that $\square \otimes_K L$ is exact we see that $\tilde{\tau} \otimes \mathrm{id}_L$ is an automorphism of $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$. Since K is in the center of $\mathcal{D}_{N,\mathcal{D}}$, one can show that the map that sends the generator $(\sum a_i t^i) \otimes l$ to the polynomial $\sum (a_i \otimes l) x^i$ is well-defined and defines a ring isomorphism from $\mathcal{D}_{N,\mathcal{D}}[t; \tilde{\tau}] \otimes_K L$ to $(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)[t; \tilde{\tau} \otimes \mathrm{id}_L]$. This isomorphism sends $t \otimes 1$ to t.

The composition leaves fixed the elements of $\mathcal{D}_{N,\mathcal{D}} \otimes_K L$, and hence sends the expression $a_0 + a_1(x \otimes 1) + \dots a_n(x \otimes 1)^n$ in S, where $a_i \in \mathcal{D}_{N,\mathcal{D}} \otimes_K L$, to the expression $a_0 + a_1 t + \dots a_n t^n$ in $(\mathcal{D}_{N,\mathcal{D}} \otimes_K L)[t; \tilde{\tau} \otimes \mathrm{id}_L]$. Therefore, the former expression equals zero if and only if $a_0 = \cdots = a_n = 0$. This finishes the proof. $\square$

We shall show at the end of the section that fields of characteristic zero always satisfy the conditions of Lemma 5.1.4. Before stating and proving the results needed to prove it, let us state a consequence of the previous lemma.

Proposition 5.1.5 (The strong algebraic eigenvalue conjecture)**.** *Let G be a countable locally indicable group and K a subfield of $\mathbb{C}$. Then, for any $\lambda \in \mathbb{C}$ which is not algebraic over K and for any $A \in \mathrm{Mat}_n(\mathcal{D}_{K[G]})$, the matrix $A - \lambda I$ is invertible in $\mathcal{U}(G)$.*

Proof. Set $L = K(\lambda^{-1})$, a subfield of $\mathbb{C}$, and let $\mathcal{D}_{K[G]}$ and $\mathcal{D}_{L[G]}$ denote the division closures of $K[G]$ and $L[G]$, respectively, in $\mathcal{U}(G)$, which are the Hughes-free division rings of fractions for $K[G]$ and $L[G]$ by Corollary 4.4.3.

Since λ is not algebraic over K the same applies to $s = \lambda^{-1}$ and hence $K[s]$ and L are isomorphic, respectively, to the polynomial ring $K[t]$ and its field of fractions $K(t)$. Thus, we can form the commutative diagram Eq. (5.2) for $\mathcal{D}_{K[G]}$ and L. In particular, $\mathcal{D}_{K[G]} \otimes_K L$ is an Ore domain and $\mathcal{Q}(\mathcal{D}_{K[G]} \otimes_K L) \cong \mathcal{D}_{K[G]}(t)$. Recall from Remark 5.1.3(2.) that this isomorphism sends $d \otimes 1 \mapsto d$ for every $d \in \mathcal{D}_{K[G]}$.

Lemma 5.1.4 tells us that $\mathcal{Q}(\mathcal{D}_{K[G]} \otimes_K L)$ is a Hughes-free division $L[G]$-ring of fractions, and hence by uniqueness (Theorem 3.4.23), it is $L[G]$-isomorphic to $\mathcal{D}_{L[G]}$. We have the following picture.

$$\begin{tikzcd}
 & & \mathcal{D}_{L[G]} \arrow[dr, "\cong"] & & \\
\mathcal{D}_{K[G]} \arrow[urr] \arrow[drr] & K[G] \arrow[l] \arrow[r] & L[G] \arrow[u] \arrow[d] & \mathcal{Q}(\mathcal{D}_{K[G]} \otimes_K L) \arrow[r, "\cong"] & \mathcal{D}_{K[G]}(t) \\
 & & \mathcal{D}_{K[G]} \otimes_K L \arrow[ur] & &
\end{tikzcd}$$

The upper left square commutes by definition, the lower left square is Eq. (5.1), and the right square is given by Hughes' theorem. Since, starting from $K[G]$, it is the same going through $\mathcal{D}_{K[G]}$ up to $\mathcal{Q}(\mathcal{D}_{K[G]} \otimes_K L)$ via $\mathcal{D}_{L[G]}$ or via $\mathcal{D}_{K[G]} \otimes_K L$, the epicity of $K[G] \to \mathcal{D}_{K[G]}$ gives us the commutativity of the outer diagram. Thus, the isomorphism $\mathcal{D}_{L[G]} \cong \mathcal{D}_{K[G]}(t)$ acts as the identity on $\mathcal{D}_{K[G]}$. Also, the commutativity of the right square and Eq. (5.2) show that $s \mapsto t$, i.e., $\lambda \mapsto t^{-1}$. Adding things up, there exists an embedding

$$\psi : \mathcal{D}_{L[G]} \hookrightarrow \mathcal{D}_{K[G]}((t)).$$

acting as the identity on $\mathcal{D}_{K[G]}$ and sending $\lambda \mapsto t^{-1}$.

Note that $(A - \lambda I_n) \in \operatorname{Mat}_n(\mathcal{D}_{L[G]})$ and that $\psi(A - \lambda I_n) = A - t^{-1}I_n$. Under the isomorphism $\operatorname{Mat}_n(\mathcal{D}_{K[G]}((t))) \cong \operatorname{Mat}_n(\mathcal{D}_{K[G]})((t))$ this matrix goes to $A - t^{-1}$, which is invertible with inverse $-\sum_{k=0}^{\infty} A^n t^{k+1}$. The injectivity of ψ implies then that $A - \lambda I_n$ must be a non-zero-divisor in the regular ring $\operatorname{Mat}_n(\mathcal{D}_{L[G]})$, and hence it must be invertible over $\operatorname{Mat}_n(\mathcal{D}_{L[G]})$, in particular as a matrix over $\mathcal{U}(G)$. □

As a remark before stating the following result, recall from Section 4.2 that if H is a subgroup of G and T is a left transversal of H in G containing the neutral element e, we identify $\mathcal{N}(H)$ as a subring of $\mathcal{N}(G)$ by letting an element $f \in \mathcal{N}(H)$ act component-wise on elements of the dense subspace $\bigoplus_{t\in T} t\ell^2(H)$ of $\ell^2(G)$, and then extending this to an element in $\mathcal{N}(G)$. On the contrary, an element $g \in G$ which is not in H does not fix (as an operator in $\mathcal{N}(G)$) the components of $\bigoplus_{t\in T} t\ell^2(H)$ (for instance, the image $(e)g = g$ of $e \in \ell^2(H)$ does not lie in $\ell^2(H)$). Therefore, we conclude that $\mathcal{N}(H) \cap G = H$. Since the elements of G define bounded operators, we also have that $\mathcal{U}(H) \cap G = H$, and consequently, for every subfield K of $\mathbb{C}$, we conclude that $\mathcal{D}_{K[H]} \cap G = H$, where as usual $\mathcal{D}_{K[H]}$ denotes the division closure of $K[H]$ in $\mathcal{U}(H)$ (or equivalently in $\mathcal{U}(G)$ since $\mathcal{U}(H)$ is regular).

The previous equality also implies that $\mathcal{D}_{\mathbb{C}[H]} \cap K^\times G = K^\times H$, for if $a = \lambda g \in \mathcal{D}_{\mathbb{C}[H]}$ for some non-zero $\lambda \in K$ then $g = \lambda^{-1}a \in \mathcal{D}_{\mathbb{C}[H]} \cap G = H$.

Proposition 5.1.6 (The center conjecture)**.** *Let G be a countable locally indicable group, K a subfield of $\mathbb{C}$ and let $\mathcal{D}_{K[G]}$ denote the division closure of $K[G]$ in $\mathcal{U}(G)$. Then*

$$\mathcal{D}_{K[G]} \cap \mathbb{C} = K.$$

Proof. Recall from Corollary 4.4.3 that $\mathcal{D}_{K[G]}$ is the Hughes-free division $K[G]$-rings of fractions and that, by Lemma 3.3.3(ii), for every $H \leq G$, the division closure of $K[H]$ in $\mathcal{D}_{K[G]}$ coincides with its division closure in $\mathcal{U}(G)$ (since $\mathcal{D}_{K[G]}$ is a division ring) and hence with $\mathcal{D}_{K[H]}$, its division closure in $\mathcal{U}(H)$. Thus, for every finitely generated subgroup $H \leq G$, $N \triangleleft H$ and $x \in H$ with $H/N = \langle Nx \rangle$ infinite cyclic, the subring of $\mathcal{D}_{K[H]}$ generated by $\mathcal{D}_{K[N]}$ and x is isomorphic to the skew polynomial ring $\mathcal{D}_{K[N]}[t; \tilde{\tau}]$, where $\tilde{\tau}$ is the automorphism of $\mathcal{D}_{K[N]}$ induced by left conjugation by x, and this gives rise to an isomorphism of $K[H]$-rings between $\mathcal{D}_{K[H]}$ and $\mathcal{D}_{K[N]}(t; \tilde{\tau})$ (see the proof of Proposition 3.4.31). Now note that the same holds for $\mathbb{C}[H]$ and that the following commutes because both isomorphisms send the element $p(t)q(t)^{-1}$ to the element $p(x)q(x)^{-1}$.

$$\begin{tikzcd}
\mathcal{D}_{K[H]} \arrow[d, hook] & \mathcal{D}_{K[N]}(t; \tilde{\tau}) \arrow[l, "\cong"'] \arrow[d, hook] \\
\mathcal{D}_{\mathbb{C}[H]} & \mathcal{D}_{\mathbb{C}[N]}(t; \tilde{\tau}) \arrow[l, "\cong"']
\end{tikzcd}$$

Since $\mathcal{D}_{K[N]}(t;\tilde{\tau})$ embeds in $\mathcal{D}_{K[N]}((t;\tilde{\tau}))$ (similarly for $\mathbb{C}[N]$), we can form a commutative diagram

$$\begin{tikzcd}
 & \mathcal{D}_{K[H]} \arrow[r, "\psi_K"] \arrow[d, hook] & \mathcal{D}_{K[N]}((t;\tilde{\tau})) \arrow[d, hook] \\
\mathbb{C} \arrow[r, hook] & \mathcal{D}_{\mathbb{C}[H]} \arrow[r, "\psi_{\mathbb{C}}"] & \mathcal{D}_{\mathbb{C}[N]}((t;\tilde{\tau}))
\end{tikzcd}$$

where ψ_K and $\psi_{\mathbb{C}}$ are injective, leave fixed the elements of $\mathcal{D}_{K[N]}$ and $\mathcal{D}_{\mathbb{C}[N]}$, respectively, and send $x \mapsto t$. The elements of $\mathbb{C}$ map through $\psi_{\mathbb{C}}$ to Laurent series in $\mathcal{D}_{\mathbb{C}[N]}((t;\tilde{\tau}))$ with just one possible non-zero summand corresponding to the (complex) constant term.

Consider the morphism of rational $K^\times G$-semirings $\Phi : \mathrm{Rat}(K^\times G) \to \mathcal{D}_{K[G]}$ and let a be a non-zero element of $\mathcal{D}_{K[G]} \cap \mathbb{C}$. If, as an element of $\mathcal{D}_{K[G]}$, $\mathrm{Tree}_G(a) = 1_{\mathcal{T}}$ and α realizes the G-complexity of a, then $\alpha \in K^\times G$ by Lemma 4.3.7(ii), and therefore $a = \Phi(\alpha) = \alpha \in K^\times G$. Thus, $a \in K^\times G \cap \mathbb{C} = K^\times$.

Assume that $\mathrm{Tree}_G(a) > 1_{\mathcal{T}}$ and let $\alpha \in \mathrm{Rat}(K^\times G)$ realize the G-complexity of a. By Theorem 4.3.8, we can express $\alpha = pu$ for some primitive element $p \in \mathrm{Rat}(K^\times G)$ and some $u \in K^\times G$, and there exists a finitely generated subgroup $\mathrm{source}(\alpha) = \mathrm{source}(p)$ of $K^\times G$ such that $\alpha \in \mathrm{Rat}(\mathrm{source}(\alpha))K^\times G$. Set $a' = \Phi(p)$ and $H = \pi_G(\mathrm{source}(p))$, where π_G denotes the composition $K^\times G \to K^\times G/K^\times = G$. Since p is primitive, this implies that $p \in \mathrm{Rat}(K^\times H)$, and hence Proposition 4.3.9 implies that $a' \in \mathcal{D}_{K[H]}$. Now $a \in \mathbb{C}$ and $(a')^{-1} \in \mathcal{D}_{K[H]}$, so their product is an element in $\mathcal{D}_{\mathbb{C}[H]}$ and we have $u = (a')^{-1}a \in \mathcal{D}_{\mathbb{C}[H]} \cap K^\times G = K^\times H$ in view of the discussion above. This means that actually $\alpha \in \mathrm{Rat}(E^\times H)$, $a \in \mathcal{D}_{K[H]}$ and α also realizes the H-complexity of a.

If H is trivial, then $a \in \mathcal{D}_K = K$. Otherwise, if $N \triangleleft H$ is such that H/N is infinite cyclic and we form the previous diagram, then $\mathcal{A} = \mathcal{D}_{K[N]}$, $\tilde{\tau}$ and $\mathcal{P} = \mathcal{D}_{K[N]}((t;\tilde{\tau}))$ satisfy the necessary conditions (i),(ii) and (iii) for applying Proposition 4.3.13. Moreover, we embed $K[H]$ in $\mathcal{D}_{K[N]}((t;\tilde{\tau}))$ via ψ_K, which coincides with the map $\tilde{\phi}$ defined in Remark 4.3.12(i). By definition of ψ_K, we also have that $\mathcal{D}_{K[H]}$ is $K[H]$-isomorphic to the division closure $\mathcal{D}_{H,\mathcal{P}} = \mathcal{D}_{K[N]}(t;\tilde{\tau})$ of $K[H]$ in $\mathcal{P}$, and then $\psi_K(a) \in \mathcal{D}_{H,\mathcal{P}}$ satisfies $\mathrm{Tree}_H(a) = \mathrm{Tree}_H(\psi_K(a))$ and α also realizes the H-complexity of $\psi_K(a)$ by Lemma 4.3.11. Finally, $\mathcal{D}_{H,\mathcal{P}}$ is a division ring, so every non-zero element is invertible, and hence we are in position to apply Proposition 4.3.13.

According to it, $\psi_K(a) = \sum a_i$ with $a_i \in \mathcal{D}_{K[N]}t^i$ and $\mathrm{Tree}_H(a_i) \leq \mathrm{Tree}_H(\psi_K(a))$. Since $a \in \mathcal{D}_{K[H]} \cap \mathbb{C}$, the commutativity of the previous diagram implies that there is only one summand in the expression as a Laurent series, namely, $\psi_K(a) = a_0$. But then $\mathrm{Tree}_H(a_0) = \mathrm{Tree}_H(\psi_K(a))$ and the same proposition states that $\alpha \in \mathrm{Rat}(K^\times N) \subseteq \mathrm{Rat}(K^\times N)K^\times G$. By Theorem 4.3.8(iv), $\mathrm{source}(\alpha) \leq K^\times N$, and hence $H \leq N$, a contradiction. This finishes the proof. $\square$

The reason why this proposition is called "the center conjecture" is the following. Observe that, in general, the center of $\mathcal{U}(G)$ coincides with the centralizer of G in $\mathcal{U}(G)$, i.e., $Z(\mathcal{U}(G)) = C_{\mathcal{U}(G)}(G)$. Indeed, since $\mathcal{U}(G)$ is the classical quotient ring of $\mathcal{N}(G)$ (Proposition 4.2.1(ii)), every $u \in C_{\mathcal{U}(G)}(G)$ can be written as $u = ab^{-1} = c^{-1}d$ for some $a, b, c, d \in \mathcal{N}(G)$. Now, $gu = ug$ for every $g \in G$, what implies that the equality $cga = dgb$

holds in $\mathcal{N}(G)$. By linearity, $cza = dzb$ for every $z \in \mathbb{C}[G]$ and by continuity, $cfa = dfb$ for every $f \in \mathcal{N}(G)$. Therefore, $fu = uf$ for every $f \in \mathcal{N}(G)$. Finally, from here u also commutes with the inverse in $\mathcal{U}(G)$ (whenever it exists) of an element in $\mathcal{N}(G)$, and hence lies in the center $Z(\mathcal{U}(G))$. This gives the equality.

Now, recall that a group G is called *ICC* if all its non-trivial conjugacy classes are infinite. For countable ICC groups, the group von Neumann algebra $\mathcal{N}(G)$ is known to be a factor, i.e., $Z(\mathcal{N}(G)) = \mathbb{C}$, and in this case the same description of the center extends to $\mathcal{U}(G)$ by [Liu12, Proposition 30], i.e., $Z(\mathcal{U}(G)) = \mathbb{C}$. The relation of the center conjecture with this fact is given in the next corollary.

Corollary 5.1.7. *Let G be a locally indicable ICC group, K a field of characteristic zero and $\mathcal{D}$ a Hughes-free division $K[G]$-ring of fractions. Then $Z(\mathcal{D}) = K$.*

Proof. Assume that there exists $a \in Z(\mathcal{D}) \setminus K$. Then there are a finitely generated subgroup $H_0 \leq G$ and a finitely generated subfield K_0 of K such that $a \in \mathcal{D}_{K_0[H_0]}$ (here $\mathcal{D}_{K_0[H_0]}$ denotes the division closure of $K_0[H_0]$ in $\mathcal{D}$). We can embed H_0 in a countable ICC subgroup H of G. Indeed, starting with H_0 we can define for every $i > 0$ a countable subgroup H_i of G such that all H_i-conjugacy classes of non-trivial elements of H_{i-1} are infinite. This can be done by adding, for every $h \in H_{i-1}$ with finite conjugacy class in H_{i-1}, a countably infinite number of elements of G defining different elements in the conjugacy class of h in G, and then taking the subgroup H_i generated by H_{i-1} and all these elements. Since H_{i-1} is countable, the number of elements chosen is also countable and hence so is H_i. Thus, $H = \bigcup H_i$ is a countable ICC group containing H_0.

By construction, $a \in Z(\mathcal{D}_{K_0[H]}) \setminus K_0$ and $\mathcal{D}_{K_0[H]}$ is a Hughes-free division $K_0[H]$-ring of fractions. For the latter claim, let H' be a non-trivial finitely generated subgroup of H, $N \triangleleft H'$ be such that H'/N is infinite cyclic and take $x \in H'$ satisfying $H'/N = \langle Nx \rangle$. Then H' is also a finitely generated subgroup of G and the Hughes-freeness of $\mathcal{D}$ implies that the powers of x are linearly independent over $\mathcal{D}_{K[N]}$, the division closure of $K[N]$ in $\mathcal{D}$. Since $\mathcal{D}_{K_0[H]}$ is a division subring of $\mathcal{D}$ and K_0 is a subfield of K, the division closure $\mathcal{D}_0$ of $K_0[N]$ in $\mathcal{D}_{K_0[H]}$ is contained in $\mathcal{D}_{K[N]}$, and hence the powers of x are in particular linearly independent over $\mathcal{D}_0$.

Let now $\varphi : K_0 \to \mathbb{C}$ be an embedding of K_0 into $\mathbb{C}$ with image $\varphi(K_0) = K_1$ and let $\varphi : K_0[H] \to K_1[H]$ denote also the induced isomorphism. Since H is locally indicable (Example 3.4.15(3)), Corollary 4.4.3 and the proof of Proposition 5.1.1 tell us that the division closure $\mathcal{D}_{K_1[H]}$ of $K_1[H]$ in $\mathcal{U}(H)$ is another Hughes-free division $K_0[H]$-ring of fractions, and hence its uniqueness (Theorem 3.4.23) implies that there exists a $K_0[H]$-isomorphism $\psi : \mathcal{D}_{K_0[H]} \to \mathcal{D}_{K_1[H]}$. In particular, we obtain that $\psi(a) \in Z(\mathcal{D}_{K_1[H]}) \setminus K_1$.

But since H is also ICC,

$$Z(\mathcal{D}_{K_1[H]}) \subseteq C_{\mathcal{D}_{K_1[H]}}(H) \subseteq C_{\mathcal{U}(H)}(H) = Z(\mathcal{U}(H)) = \mathbb{C},$$

and hence $\psi(a)$ is a non-zero element in $\mathcal{D}_{K_1[H]} \cap \mathbb{C}$ which is not in K_1, a contradiction by Proposition 5.1.6. Thus, such an a cannot exist.

Since $K \subseteq Z(\mathcal{D})$ by Corollary 4.1.12 we deduce that $Z(\mathcal{D}) = K$. $\square$

We can finally use this result to prove that fields of characteristic zero satisfy the hypothesis of Lemma 5.1.4.

Corollary 5.1.8. *Let L/K be an extension of fields of characteristic zero, G a locally indicable group and $\mathcal{D}$ a Hughes-free division $K[G]$-ring of fractions. Then $\mathcal{D} \otimes_K L$ is a domain. In particular, the Hughes-free division $L[G]$-ring of fractions is isomorphic to the Ore division ring $\mathcal{Q}(\mathcal{D} \otimes_K L)$.*

Proof. First let us assume that G is ICC. In this case $Z(\mathcal{D}) = K$ by Corollary 5.1.7, and therefore $\mathcal{D} \otimes_K L$ is a simple ring by Remark 5.1.3(4.). By Corollary 5.1.2, there exists a Hughes-free division $L[G]$-ring of fractions $\tilde{\mathcal{D}}$, and we can identify $\mathcal{D}$ with the division closure of $K[G]$ in $\tilde{\mathcal{D}}$. The K-biadditive map $\mathcal{D} \times L \to \tilde{\mathcal{D}}$ given by $(d, l) \to dl$ gives rise to a ring homomorphism $\mathcal{D} \otimes_K L \to \tilde{\mathcal{D}}$, and this map must be injective because $\mathcal{D} \otimes_K L$ is simple. Therefore, $\mathcal{D} \otimes_K L$ is isomorphic to a subring of $\tilde{\mathcal{D}}$ (the subring generated by $\mathcal{D}$ and L), and hence a domain.

For an arbitrary G, the restricted standard wreath product $G \wr \mathbb{Z}$ is again locally indicable (Example 3.4.15(8.)) and ICC ([Pré13, Corollary 4.2]), and so $\mathcal{D} \otimes_K L$ can be embedded in the domain $\mathcal{D}_{G\wr\mathbb{Z}} \otimes_K L$, where $\mathcal{D}_{G\wr\mathbb{Z}}$ denotes the Hughes-free division $K[G \wr \mathbb{Z}]$-ring of fractions. This concludes the proof, and the last assertion follows from Lemma 5.1.4. □

If K and L are subfields of $\mathbb{C}$, this corollary states that the division closure $\mathcal{D}_{L[G]}$ of $L[G]$ inside $\mathcal{U}(G)$ is isomorphic to the classical quotient ring of $\mathcal{D}_{K[G]} \otimes_K L$. It is proved throughout [Jai19] that the same statement holds for sofic groups satisfying the strong Atiyah conjecture. We expect this property to hold in general.

Conjecture. *Let L/K be an extension of subfields of $\mathbb{C}$ and G any (countable) group satisfying the strong Atiyah conjecture. Let $\mathcal{D}_{K[G]}$ and $\mathcal{D}_{L[G]}$ denote, respectively, the division closures of $K[G]$ and $L[G]$ in $\mathcal{U}(G)$. Then $\mathcal{D}_{L[G]}$ is isomorphic to the classical quotient ring of $\mathcal{D}_{K[G]} \otimes_K L$.*

5.2 Lück's approximation conjecture in the space of marked groups

In this section we exploit the methods developed in Section 4.3.3 and Section 4.4 to prove Lück's approximation conjecture in the space of marked groups when the group being approximated is virtually locally indicable. This result relies heavily on the capability to compare between different Sylvester matrix rank functions. For this purpose, we start with a subsection that gathers some general results on the subject.

Before starting the first subsection, let us make a general observation that shall be frequently used in the following. Assume that we have an isomorphism of groups $H \cong H'$. This gives rise to a $*$-isomorphism $\phi : \mathbb{C}[H] \to \mathbb{C}[H']$, and if H is countable, this extends to $*$-isomorphisms $\phi : \mathcal{N}(H) \to \mathcal{N}(H')$ and $\phi : \mathcal{U}(H) \to \mathcal{U}(H')$ (in the same way we did in Section 4.2). Now ϕ preserves the trace of elements of $\mathcal{N}(H)$ ([Lüc02, Lemma 1.24(i)]) and sends projections to projections, so that we can see from the definition of rk_H on

$\mathcal{U}(H)$ that, for every $a \in \mathcal{U}(H)$, $\operatorname{rk}_H(a) = \operatorname{rk}_{H'}(\phi(a))$. Thus, $\phi^\sharp(\operatorname{rk}_{H'}) = \operatorname{rk}_H$ as rank functions on $\mathcal{U}(H)$.

In particular, if $H' \leq G$ and we denote by $\iota : \mathcal{U}(H') \to \mathcal{U}(G)$ the natural embedding, then by Proposition 4.2.2, $\operatorname{rk}_H = \phi^\sharp(\iota^\sharp(\operatorname{rk}_G))$ as rank functions on $\mathcal{U}(H)$, and hence on $\mathbb{C}[H]$.

5.2.1 Comparing Sylvester matrix rank functions

Recall that, given two Sylvester matrix rank functions rk_1 and rk_2 on a ring R, we write $\operatorname{rk}_1 \leq \operatorname{rk}_2$ if, for every matrix A over R, $\operatorname{rk}_1(A) \leq \operatorname{rk}_2(A)$.

In the spirit of the definition of regular (resp. $*$-regular, epic division) envelope of a regular (resp. $*$-regular, integer-valued) Sylvester matrix rank function, given a Sylvester matrix rank function rk on a ring R, we will call *envelope* of rk to a triple $(S, \varphi, \operatorname{rk}')$ such that $\varphi : R \to S$ is a ring homomorphism, rk' is a faithful rank function on S and $\varphi^\sharp(\operatorname{rk}') = \operatorname{rk}$. Observe that by Lemma 1.3.11(b) rk induces a faithful rank function on $R/\ker \operatorname{rk}$, and hence an envelope of rk always exists.

Proposition 5.2.1. *Let R be a ring and let $\{\operatorname{rk}_i\}_{i=1}^n$ be a family of Sylvester matrix rank functions on R such that $\operatorname{rk}_i \leq \operatorname{rk}_{i+1}$ for every i. Assume that $(S_i, \varphi_i, \operatorname{rk}_i')$ is an envelope of rk_i, and set $S = \prod S_i$, $\varphi = (\varphi_i) : R \to S$. If $\mathcal{D}$ is the division closure of $\varphi(R)$ in S and $\pi_i : \mathcal{D} \to S_i$ is the restriction of the standard projection, then*

$$\pi_i^\sharp(\operatorname{rk}_i') \leq \pi_{i+1}^\sharp(\operatorname{rk}_{i+1}')$$

as rank functions on $\mathcal{D}$. In particular, π_n is injective.

Proof. Let A be a matrix over $\mathcal{D}$. Recall from Remark 3.3.7 that $\mathcal{D}$ is contained in the rational closure $\mathcal{R}^\varphi(S)$, and hence by Cramer's rule Proposition 3.3.8, there exist $k \geq 1$, a matrix A' over $\varphi(R)$ and invertible matrices P, Q over $\mathcal{R}^\varphi(S)$ (in particular over S) such that

$$I_k \oplus A = PA'Q.$$

Since $\pi_i^\sharp(\operatorname{rk}_i')$, as a rank function on $\mathcal{D}$, is the restriction of the corresponding rank function on S, we obtain that, for every i,

$$\pi_i^\sharp(\operatorname{rk}_i')(A) + k = \pi_i^\sharp(\operatorname{rk}_i')(A').$$

If $A' = \varphi(B)$ for some $B \in \operatorname{Mat}(R)$, taking into account that $\pi_i \circ \varphi = \varphi_i$ and that, by definition, $\operatorname{rk}_i = \varphi_i^\sharp(\operatorname{rk}_i')$, the right hand side becomes $\operatorname{rk}_i'(\pi_i(A')) = \operatorname{rk}_i'(\varphi_i(B)) = \operatorname{rk}_i(B)$. Therefore, for every $i = 1, \dots, n-1$,

$$\pi_i^\sharp(\operatorname{rk}_i')(A) = \operatorname{rk}_i(B) - k \leq \operatorname{rk}_{i+1}(B) - k = \pi_{i+1}^\sharp(\operatorname{rk}_{i+1}')(A),$$

what gives the first assertion of the proposition. Finally, if $d = (d_1, \dots, d_n)$ is an element of $\mathcal{D}$ such that $d_n = \pi_n(d) = 0$, then for every $i \leq n$,

$$\operatorname{rk}_i'(\pi_i(d)) \leq \operatorname{rk}_n'(\pi_n(d)) = 0,$$

from where the faithfulness of rk_i' implies that $d_i = \pi_i(d) = 0$, i.e., $d = 0$. $\square$

Let us add now a bit of notation: If we have a ring R and a family of R-rings (S_i, φ_i) for $i = 1, \dots, n$, then we shall use subscripts to refer to the cartesian product of the elements in the family corresponding to these subscripts, and every construction regarding them. For instance, by S_{13} we mean $S_1 \times S_3$ and by φ_{13} the map $\varphi_{13} : (\varphi_1, \varphi_3) : R \to S_{13}$. In a similar way, $\mathcal{D}_{R,13}$ shall denote the division closure of $\varphi_{13}(R)$ in S_{13}. We shall denote exceptionally $S := S_{1\dots n}$, $\varphi := \varphi_{1\dots n}$, $\mathcal{D}_{R,S} = \mathcal{D}_{R,1\dots n}$, and again we will use subscripts to denote the projections onto the chosen cartesian products, so that π_{13} denotes the standard projection $\pi_{13} : S \to S_{13}$. The following commutes for every choice of $i_1 \leq \cdots \leq i_k$,

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\varphi_{i_1\dots i_k}"'] & S \arrow[d, "\pi_{i_1\dots i_k}"] \\ & S_{i_1\dots i_k}, \end{tikzcd}$$

Notice that, in general, if $\psi : S \to S'$ is a ring homomorphism and R is a subring of S, then $\psi(\mathcal{D}_{R,S}) \subseteq \mathcal{D}_{\psi(R),S'}$, what can be proved as in Lemma 3.3.3(3). Thus, the restriction of $\pi_{i_1\dots i_k}$ to $\mathcal{D}_{R,S}$ is actually a map $\pi_{i_1\dots i_k} : \mathcal{D}_{R,S} \to \mathcal{D}_{R,i_1,\dots,i_k}$.

Assume now that $S_i := \mathcal{U}_i$ is regular for every i (hence $S := \mathcal{U}$ is regular) and $\{\mathrm{rk}_i\}_{i=1}^n$ is a family of regular Sylvester matrix rank functions on R satisfying $\mathrm{rk}_1 \leq \cdots \leq \mathrm{rk}_n$ and with regular envelopes $(\mathcal{U}_i, \varphi_i, \mathrm{rk}'_i)$.

In this event, let $d = (d_1, \dots, d_n)$ be an element of $\mathcal{D}_{R,\mathcal{U}}$, and observe that d is invertible in $\mathcal{U}$ (and hence in $\mathcal{D}_{R,\mathcal{U}}$ since it is division closed) if and only if every component is invertible in $\mathcal{U}_i$. In fact, if d_1 is invertible in $\mathcal{U}_1$ then, for every $i \geq 1$, Proposition 5.2.1 tells us that $\mathrm{rk}'_i(d_i) \geq \mathrm{rk}'_1(d_1) = 1$, i.e., $\mathrm{rk}'_i(d_i) = 1$. Regularity of $\mathcal{U}_i$ and faithfulness of rk'_i imply that d_i is invertible for every $i \geq 1$ (Lemma 1.3.12). In other words, we have seen that d is invertible if and only if d_1 is invertible in $\mathcal{U}_1$.

A similar argument shows that, in general, an element in $\mathcal{D}_{1i_2\dots i_k}$ is invertible if and only if its first component is invertible in $\mathcal{U}_1$, and therefore an element $d \in \mathcal{D}_{R,\mathcal{U}}$ is invertible if and only if $\pi_{1i_2\dots i_k}(d)$ is invertible in $\mathcal{D}_{1i_2\dots i_k}$. This has two consequences:

- The map $\pi_{1i_2\dots i_k} : \mathcal{D}_{R,\mathcal{U}} \to \mathcal{D}_{1i_2\dots i_k}$ is surjective. Indeed, take any element $d = (d_1, \dots, d_n) \in \mathcal{D}_{R,\mathcal{U}}$ such that $\pi_{1i_2\dots i_k}(d) = (d_1, d_{i_2}, \dots, d_{i_k})$ is invertible in $\mathcal{D}_{1i_2,\dots i_k}$. We have shown that d is then invertible in $\mathcal{D}$ and $\pi_{1i_2\dots i_k}(d^{-1}) = \pi_{1i_2\dots i_k}(d)^{-1}$, what shows that $\pi_{1i_2\dots i_k}(\mathcal{D}_{R,\mathcal{U}})$ is a division closed subring of $\mathcal{D}_{1i_2\dots i_k}$ containing $\varphi_{1i_2,\dots,i_k}(R)$. Thus $\pi_{1i_2\dots i_k}(\mathcal{D}_{R,\mathcal{U}}) = \mathcal{D}_{1i_2\dots i_k}$, as claimed.

- If in the previous setting $R = E * G$ for some group G and some division ring E, then $\pi_{1i_2\dots i_k}$ is a morphism of rational $E^\times G$-semirings, since we have just shown that it preserves the $\diamond$-operation.

We use these observations and notation in the following two rather technical consequences of the previous result, which are essential for the proof of the Lück approximation conjecture in the next subsection.

Corollary 5.2.2. *Let G be a group, E a division ring and $\mathrm{rk}_1, \mathrm{rk}_2$ be regular Sylvester matrix rank functions on $R = E * G$ with regular envelopes $(\mathcal{U}_i, \mathrm{rk}'_i, \varphi_i)$, $i = 1, 2$. If $\mathrm{rk}_1 \leq \mathrm{rk}_2$, the following diagram commutes*

$$\begin{tikzcd} \mathrm{Rat}(E^\times G) \arrow[r, "\Phi_{\mathcal{U}}"] \arrow[dr, "\Phi_1"'] & \mathcal{D}_{R,\mathcal{U}} \arrow[d, "\pi_1"] \\ & \mathcal{D}_{R,1}, \end{tikzcd}$$

where $\Phi_{\mathcal{U}}$ and Φ_1 are the unique morphisms of rational $E^\times G$-semirings.

Proof. The previous observations show that π_1, and hence $\pi_1 \circ \Phi_{\mathcal{U}}$, are morphisms of $E^\times G$-semirings, from where the uniqueness of Φ_1 implies that $\Phi_1 = \pi_1 \circ \Phi_{\mathcal{U}}$. □

Corollary 5.2.3. *Let H be a group, E a division ring and $\mathrm{rk}_1, \mathrm{rk}_2, \mathrm{rk}_3$ regular Sylvester matrix rank functions on $R = E * H$ with regular envelopes $(\mathcal{U}_i, \mathrm{rk}'_i, \varphi_i)$, $i = 1, 2, 3$. Assume that $\mathrm{rk}_1 \leq \mathrm{rk}_2 \leq \mathrm{rk}_3$ and consider the universal morphisms*

$$\Phi_{12} : \mathrm{Rat}(E^\times H) \to \mathcal{D}_{R,12} \quad \Phi_{13} : \mathrm{Rat}(E^\times H) \to \mathcal{D}_{R,13}.$$

Consider any $\alpha \in \mathrm{Rat}(E^\times H)$. If $\Phi_{12}(\alpha)$ is non-zero then $\Phi_{13}(\alpha)$ is non-zero. Moreover, $\Phi_{12}(\alpha)$ is invertible if and only if $\Phi_{13}(\alpha)$ is invertible.

Proof. Let $\Phi_{\mathcal{U}} : \mathrm{Rat}(E^\times H) \to \mathcal{D}_{R,\mathcal{U}}$ be the unique morphism of $E^\times G$-semirings. The previous observations imply that π_{12} and π_{23} are morphisms of rational $E^\times G$-semirings, so as in Corollary 5.2.2, the following commutes.

$$\begin{tikzcd} & \mathcal{D}_{R,\mathcal{U}} \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{13}"] & \\ \mathcal{D}_{R,12} & \mathrm{Rat}(E^\times H) \arrow[l, "\Phi_{12}"'] \arrow[r, "\Phi_{13}"] \arrow[u, "\Phi_{\mathcal{U}}"'] & \mathcal{D}_{R,13} \end{tikzcd}$$

Take any $\alpha \in \mathrm{Rat}(E^\times H)$ and set $\Phi_{\mathcal{U}}(\alpha) = (d_1, d_2, d_3)$. Hence, we have that

$$\Phi_{12}(\alpha) = (d_1, d_2), \quad \Phi_{13}(\alpha) = (d_1, d_3).$$

As in the previous observations, $\Phi_{12}(\alpha)$ is invertible if and only if d_1 is invertible if and only if $\Phi_{13}(\alpha)$ is invertible. Finally, since Proposition 5.2.1 gives $\mathrm{rk}'_1(d_1) \leq \mathrm{rk}'_2(d_2) \leq \mathrm{rk}'_3(d_3)$, the faithfulness of each rk'_i implies that if $\Phi_{12}(\alpha)$ is non-zero, then $\mathrm{rk}'_3(d_3) > 0$, and hence $\Phi_{13}(\alpha)$ is non-zero. □

The motivation and necessity of this result lie in the fact that, at a certain point, we will need to prove that an element, expressible as $\Phi_{12}(\alpha)$, is invertible. Corollary 5.2.3 will then allow us to pass from the ambient $\mathcal{U}_1 \times \mathcal{U}_2$ to an appropriate ambient $\mathcal{P} = \mathcal{U}_1 \times \mathcal{U}_3$ on which the conditions of Proposition 4.3.13 are satisfied, and therefore, to tackle instead the invertibility of the non-zero element $\Phi_{13}(\alpha)$ by means of induction on the complexity.

We are now going to give another proof of [JL20, Lemma 7.7] by using a slightly more general but slightly less powerful argument than that of [JL20, Lemma 7.6] but which does not require regularity of the ring.

Lemma 5.2.4. *Let R be a ring with a Sylvester matrix rank function* rk *with natural transcendental extension* $\widetilde{\mathrm{rk}}$ *to $R[t]$. If $\mathfrak{e} : R[t] \to R$ is the evaluation at 1_R, then* $\mathfrak{e}^\sharp(\mathrm{rk}) \leq \widetilde{\mathrm{rk}}$.

Proof. Note that $\mathfrak{e}$ is actually a surjective ring homomorphism since we evaluate at a central element. We want to prove that for every matrix A over $R[t]$, $\mathrm{rk}(\mathfrak{e}(A)) \leq \widetilde{\mathrm{rk}}(A)$ and note that if $A = (p_{ij}(t))$, the element in the ij-position in $\mathfrak{e}(A)$ is just the sum of the coefficients in p_{ij}. On the other hand $\widetilde{\mathrm{rk}}(A)$ can be computed as

$$\widetilde{\mathrm{rk}}(A) = \lim_{k\to\infty} \frac{\mathrm{rk}(\psi_k(A))}{k},$$

where $\psi_k : R[t] \to \mathrm{Mat}_k(R)$ maps p to the matrix associated to ϕ_k^p, the endomorphism of $R[t]/R[t]t^k$ given by right multiplication by p, with respect to the canonical bases. Let us illustrate the proof for an $(n = 1) \times (m = 2)$ matrix $A = (a(t), b(t))$, with $a(t) = a_0 + a_1 t + a_2 t^2$, $b(t) = b_1 t + b_3 t^3$ while describing the procedure for a general matrix.

The larger the k is taken (larger than the maximum of the degrees of the $p_{ij}(t)$) the larger the number of rows in $\psi_k(A)$ in which we can see every coefficient of $p_{ij}(t)$. For instance, if $k = 7$, and since there is no twisting involved, we obtain (see Eq. (1.2))

$$\psi_7(A) = \left[\begin{array}{ccccccc|ccccccc}
a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 0 & 0 \\
0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 \\
0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 \\
\hline
0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 \\
0 & 0 & 0 & 0 & 0 & a_0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 \\
0 & 0 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Let us denote by d the maximum degree among the polynomials in A (in our case, $d = 3$). We are going to delete the last d rows of each of the n horizontal blocks of A (in our case, A has just $n = 1$ horizontal blocks, and we are going to delete the last $d = 3$ rows, the ones below the red line). After doing this, we pass from the $nk \times mk$ matrix $\psi_k(A)$ to an $n(k-d) \times mk$ matrix B_k (in our case B_k has size $1(7-3) \times 2 \cdot 7 = 4 \times 14$). Observe that in general, since d is fixed from the beginning, we have by the properties of rank functions that

$$\mathrm{rk}(B_k) \leq \mathrm{rk}(\psi_k(A)) \leq \mathrm{rk}(B_k) + nd$$

and hence $\displaystyle\lim_{k\to\infty} \frac{\mathrm{rk}(\psi_k(A)) - \mathrm{rk}(B_k)}{k} = 0$, from where $\displaystyle\widetilde{\mathrm{rk}}(A) = \lim_{k\to\infty} \frac{\mathrm{rk}(B_k)}{k}$. In our example,

$$B_k = \left[\begin{array}{ccccccc|ccccccc}
a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 0 & 0 \\
0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 \\
0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3
\end{array}\right]$$

We are going to sum and interchange rows and columns, what leaves the rank unchanged. In the first place, we add to the first column of every vertical block the rest of the columns of that block. Doing this, we obtain in that column the sum of coefficients of the polynomial. In our case,

$$\left[\begin{array}{ccccccc|ccccccc} a_0+a_1+a_2 & a_1 & a_2 & 0 & 0 & 0 & 0 & b_1+b_3 & b_1 & 0 & b_3 & 0 & 0 & 0 \\ a_0+a_1+a_2 & a_0 & a_1 & a_2 & 0 & 0 & 0 & b_1+b_3 & 0 & b_1 & 0 & b_3 & 0 & 0 \\ a_0+a_1+a_2 & 0 & a_0 & a_1 & a_2 & 0 & 0 & b_1+b_3 & 0 & 0 & b_1 & 0 & b_3 & 0 \\ a_0+a_1+a_2 & 0 & 0 & a_0 & a_1 & a_2 & 0 & b_1+b_3 & 0 & 0 & 0 & b_1 & 0 & b_3 \end{array}\right]$$

Now, in each horizontal block we do the same, namely, starting from the bottom row in the block, we substract row $i-1$ to row i in the block from $i=(k-d),\dots,2$. In the example, we take the fourth row and substract the third, then we substract the second to the third and finally the first to the second. We get to

$$\left[\begin{array}{ccccccc|ccccccc} a_0+a_1+a_2 & a_1 & a_2 & 0 & 0 & 0 & 0 & b_1+b_3 & b_1 & 0 & b_3 & 0 & 0 & 0 \\ 0 & a_0-a_1 & a_1-a_2 & a_2 & 0 & 0 & 0 & 0 & -b_1 & b_1 & -b_3 & b_3 & 0 & 0 \\ 0 & -a_0 & a_0-a_1 & a_1-a_2 & a_2 & 0 & 0 & 0 & 0 & -b_1 & b_1 & -b_3 & b_3 & 0 \\ 0 & 0 & -a_0 & a_0-a_1 & a_1-a_2 & a_2 & 0 & 0 & 0 & 0 & -b_1 & b_1 & -b_3 & b_3 \end{array}\right]$$

The next step goes as follows: for each vertical block, we add to each column, starting from the second in the block, the rest of the columns in the same block. For instance, in our case, we add columns three to seven to the second column, then column four to seven to the third and so on. And then we do the same in the blocks of b's. We then arrive to the matrix

$$\left[\begin{array}{ccccccc|ccccccc} a_0+a_1+a_2 & a_1+a_2 & a_2 & 0 & 0 & 0 & 0 & b_1+b_3 & b_1+b_3 & b_3 & b_3 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 \end{array}\right]$$

By performing several column interchanges respecting the order of the rest of the columns, we can put together the first columns of every block. In our case,

$$\left[\begin{array}{cc|cccccccccccc} a_0+a_1+a_2 & b_1+b_3 & a_1+a_2 & a_2 & 0 & 0 & 0 & 0 & b_1+b_3 & b_3 & b_3 & 0 & 0 & 0 \\ \hline 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & 0 & 0 & b_1 & 0 & b_3 \end{array}\right]$$

The next step would be to do the same with the first row of every block. In the end, we arrive to a matrix in which the first $n\times m$ submatrix is formed by the sum of the coefficients of the polynomials in A, below this block everything is zero, and in the lower right corner we have B_{k-1}. Using then property (SMat4) of Sylvester matrix rank functions,

$$\operatorname{rk}(B_k)\geq \operatorname{rk}(\mathfrak{e}(A))+\operatorname{rk}(B_{k-1})$$

Inductively we can see that for every $k > d$, $\mathrm{rk}(B_k) \geq (k-d)\mathrm{rk}(\mathfrak{e}(A))$ (in our case, $\mathrm{rk}(B_7) \geq 4\mathrm{rk}(\mathfrak{e}(A))$). Therefore, in the limit,

$$\widetilde{\mathrm{rk}}(A) = \lim_{k\to\infty} \frac{\mathrm{rk}(B_k)}{k} \geq \lim_{k\to\infty} \frac{(k-d)\mathrm{rk}(\mathfrak{e}(A))}{k} = \mathrm{rk}(\mathfrak{e}(A)).$$

The procedure is valid for every matrix, and hence we obtain the desired result. □

Note that since 1_R is a central unit, the evaluation at 1_R defines also a homomorphism $\mathfrak{e} : R[t^{\pm 1}] \to R$, and the result holds for $R[t^{\pm 1}]$ because, for every matrix A over $R[t^{\pm 1}]$, there exists $k \geq 0$ such that $A(t^k I)$ is a matrix over $R[t]$. Since $t^k I$ is invertible and $\mathfrak{e}(A(t^k I)) = \mathfrak{e}(A)$, we obtain that

$$\mathrm{rk}(\mathfrak{e}(A)) = \mathrm{rk}(\mathfrak{e}(A(t^k I))) \leq \widetilde{\mathrm{rk}}(A(t^k I)) = \widetilde{\mathrm{rk}}(A).$$

Corollary 5.2.5. *Let H be a countable group and consider $G = H \times \mathbb{Z}$. If $\pi : \mathbb{C}[G] \to \mathbb{C}[H]$ is the homomorphism induced by the projection $G \to H$, then $\pi^\sharp(\mathrm{rk}_H) \leq \mathrm{rk}_G$.*

Proof. Let us write things carefully this time. Using multiplicative notation for $\mathbb{Z}$, the isomorphism $\phi : H \cong H \times \{1\} \leq G$ gives rise to an isomorphism $\tilde{\phi} : \mathbb{C}[H] \to \mathbb{C}[\phi(H)]$, and $\mathrm{rk}_H = \tilde{\phi}^\sharp(\mathrm{rk}_{\phi(H)})$. Now $G/\phi(H) \cong \mathbb{Z}$, there exists an isomorphism $\psi : \mathbb{C}[G] \cong \mathbb{C}[\phi(H)][t^{\pm 1}]$ and Proposition 4.2.7 tells us that rk_G is the natural extension of $\mathrm{rk}_{\phi(H)}$, meaning that $\mathrm{rk}_G = \psi^\sharp(\widetilde{\mathrm{rk}}_{\phi(H)})$.

If $\mathfrak{e} : \mathbb{C}[\phi(H)][t^{\pm 1}] \to \mathbb{C}[\phi(H)]$ is the evaluation at 1, what defines a homomorphism since 1 is a central unit, then the following commutes

$$\begin{tikzcd}
\mathbb{C}[G] \arrow[r, "\pi"] \arrow[d, "\psi"', "\cong"] & \mathbb{C}[H] \arrow[d, "\tilde{\phi}", "\cong"'] \\
\mathbb{C}[\phi(H)][t^{\pm 1}] \arrow[r, "\mathfrak{e}"] & \mathbb{C}[\phi(H)].
\end{tikzcd}$$

Lemma 5.2.4 (and the subsequent discussion) tells us that, as ranks on $\mathbb{C}[\phi(H)][t^{\pm 1}]$, we have $\mathfrak{e}^\sharp(\mathrm{rk}_{\phi(H)}) \leq \widetilde{\mathrm{rk}}_{\phi(H)}$. Therefore,

$$\pi^\sharp(\mathrm{rk}_H) = \pi^\sharp\tilde{\phi}^\sharp(\mathrm{rk}_{\phi(H)}) = \psi^\sharp\mathfrak{e}^\sharp(\mathrm{rk}_{\phi(H)}) \leq \psi^\sharp(\widetilde{\mathrm{rk}}_{\phi(H)}) = \mathrm{rk}_G,$$

as we wanted to show. □

When rk takes integer values, a much stronger result than Lemma 5.2.4 holds even for skew Laurent polynomials.

Lemma 5.2.6. *Let R be a ring and τ an automorphism of R. Set $S = R[t^{\pm 1};\tau]$ and let* rk *be a τ-compatible integer-valued Sylvester matrix rank function on R. Then the natural extension* $\widetilde{\mathrm{rk}}$ *of* rk *is universal among the Sylvester matrix rank functions on S that extend* rk.

Proof. Let $(\mathcal{D}, \phi)$ be the division envelope of rk. By Proposition 3.1.20, τ and ϕ extend, respectively, to an automorphism $\tilde{\tau}$ of $\mathcal{D}$ such that $\tilde{\tau} \circ \phi = \phi \circ \tau$ and to a homomorphism $\tilde{\phi} : S \to \mathcal{D}[t^{\pm 1}; \tilde{\tau}]$ that acts as ϕ on R and sends $t \mapsto t$. Moreover, the division envelope of $\widetilde{\mathrm{rk}}$ is the division ring $\mathcal{D}(t; \tilde{\tau})$ together with the composition $S \to \mathcal{D}[t^{\pm 1}; \tilde{\tau}] \to \mathcal{D}(t; \tilde{\tau})$. If ι denotes the latter embedding, then observe that $\iota^\sharp(\mathrm{rk}_{\mathcal{D}(t;\tilde{\tau})})$ is universal in $\mathbb{P}(\mathcal{D}[t^{\pm 1}; \tilde{\tau}])$.

Indeed, since any matrix A over $\mathcal{D}[t^{\pm 1}; \tilde{\tau}]$ can be written as a product $A = PDQ$, where P and Q are invertible over $\mathcal{D}[t^{\pm 1}; \tilde{\tau}]$ and D is diagonal, and since $\iota^\sharp(\mathrm{rk}_{\mathcal{D}(t;\tilde{\tau})})$ gives rank 1 to every non-zero element, the rank of A will be the number of non-zero entries in D, which is the maximum value it can take. We will show that if rk' is a rank function on S extending rk, then there exists a rank rk_0 on $\mathcal{D}[t^{\pm 1}; \tilde{\tau}]$ with $\mathrm{rk}' = \tilde{\phi}^\sharp(\mathrm{rk}_0)$. In this event, $\mathrm{rk}' \leq \tilde{\phi}^\sharp(\iota^\sharp(\mathrm{rk}_{\mathcal{D}(t;\tilde{\tau})})) = \widetilde{\mathrm{rk}}$ and the proof is finished.

Let rk' be such a function on S, let Σ denote the set of square matrices over R with maximum rk-rank, and consider the localizations R_Σ and S_Σ of R and S at Σ, respectively. Since rk is τ-compatible, $\tau(\Sigma) = \Sigma$, and hence we have the following commutative diagram

$$\begin{tikzcd} R \arrow[r, "\tau"] \arrow[d, "\lambda"'] & R \arrow[r, "i"] \arrow[d, "\lambda"] & S \arrow[d, "\mu"] \\ R_\Sigma \arrow[r, "\bar{\tau}"'] & R_\Sigma \arrow[r, "f"'] & S_\Sigma \end{tikzcd}$$

Here, λ and μ denote the universal Σ-inverting maps, i is the inclusion, the existence of $\bar{\tau}$ and f is a consequence of the fact that $\lambda \circ \tau$ and $\mu \circ i$ are Σ-inverting, and $\bar{\tau}$ is an automorphism of R_Σ since τ is an automorphism of R.

The commutativity of the left square allows us to extend λ to a homomorphism $\tilde{\lambda} : S \to R_\Sigma[t^{\pm 1}; \bar{\tau}]$ that acts as λ on R and sends $t \mapsto t$, and we are going to show that $R_\Sigma[t^{\pm 1}; \bar{\tau}] \cong S_\Sigma$.

- On the hand, observe that $\tilde{\lambda}$ is Σ-inverting, and hence there exists a homomorphism $\tilde{f} : S_\Sigma \to R_\Sigma[t^{\pm 1}; \bar{\tau}]$ such that $\tilde{f} \circ \mu = \tilde{\lambda}$.

- On the other hand, we are going to show that f gives rise to a map $\tilde{f}' : R_\Sigma[t^{\pm 1}; \bar{\tau}] \to S_\Sigma$ by using the universal property of skew Laurent polynomials (cf. [GW04, Exercise 1N]). For this, we are going to show that the unit $\mu(t) \in S_\Sigma$ satisfies $\mu(t)f(a) = f\bar{\tau}(a)\mu(t)$ for every $a \in R_\Sigma$. Let $c_{\mu(t)}$ denote the automorphism of S_Σ given by left conjugation by $\mu(t)$, and note that for every r in R, the commutativity of the previous diagram gives

$$\begin{aligned} (c_{\mu(t)} \circ f \circ \lambda)(r) &= c_{\mu(t)}(\mu \circ i(r)) = \mu(t)\mu(i(r))\mu(t)^{-1} = \mu(ti(r)t^{-1}) \\ &= \mu(i\tau(r)) = (f \circ \bar{\tau} \circ \lambda)(r). \end{aligned}$$

  The epicity of λ implies then that $c_{\mu(t)} \circ f = f \circ \bar{\tau}$, i.e., that for every a in R_Σ, $\mu(t)f(a)\mu(t)^{-1} = f\bar{\tau}(a)$, as we wanted to show. The induced map $\tilde{f}'$ then acts as f on R_Σ and sends $t \mapsto \mu(t)$. If $j : R_\Sigma \to R_\Sigma[t^{\pm 1}; \bar{\tau}]$ is the inclusion, we are saying that $\tilde{f}' \circ j = f$.

- $\tilde{f}$ and $\tilde{f}'$ are mutual inverses. Let us show first that $\tilde{f}\circ\tilde{f}' = \mathrm{id}_{R_\Sigma[t^{\pm1};\bar{\tau}]}$. Note that
$$\tilde{f}\circ\tilde{f}'\circ j\circ\lambda = \tilde{f}\circ f\circ\lambda = \tilde{f}\circ\mu\circ i = \tilde{\lambda}\circ i = j\circ\lambda,$$
what by the epicity of λ implies $\tilde{f}\circ\tilde{f}'\circ j = j$, i.e., $\tilde{f}\circ\tilde{f}'$ acts as the identity on elements of R_Σ. Together with the fact that $\tilde{f}\circ\tilde{f}'(t) = \tilde{f}(\mu(t)) = \tilde{\lambda}(t) = t$, this means $\tilde{f}\circ\tilde{f}' = \mathrm{id}_{R_\Sigma[t^{\pm1};\bar{\tau}]}$.

  To show that $\tilde{f}'\circ\tilde{f} = \mathrm{id}_{S_\Sigma}$ it suffices to show by epicity that they are equal when precomposing with μ. Observe that
$$\tilde{f}'\circ\tilde{f}\circ\mu\circ i = \tilde{f}'\circ\tilde{\lambda}\circ i = \tilde{f}'\circ j\circ\lambda = f\circ\lambda = \mu\circ i,$$
what means that $\tilde{f}'\circ\tilde{f}\circ\mu$ coincides with μ on elements of R. Finally, since $(\tilde{f}'\circ\tilde{f}\circ\mu)(t) = \tilde{f}'(\tilde{\lambda}(t)) = \tilde{f}'(t) = \mu(t)$, we obtain the desired equality $\tilde{f}'\circ\tilde{f}\circ\mu = \mu$.

Now, Proposition 3.3.10 tells us that $\mathrm{rk}\in\mathrm{im}\,\lambda^\sharp$ and $\mathrm{rk}'\in\mathrm{im}\,\mu^\sharp$ (rk' coincides with rk on matrices over R and hence gives maximum rank to the elements in Σ), i.e., there exist Sylvester matrix rank functions rk_Σ and rk'_Σ on R_Σ and S_Σ, respectively, such that $\mathrm{rk} = \lambda^\sharp(\mathrm{rk}_\Sigma)$ and $\mathrm{rk}' = \mu^\sharp(\mathrm{rk}'_\Sigma)$. Moreover, by Cramer's rule rk_Σ is integer-valued.

If we set $I = \ker\mathrm{rk}_\Sigma$, then the proof of [Sch85, Theorem 7.5] tells us that I is the maximal ideal of R_Σ and that R_Σ/I is a division ring. Furthermore, rk_Σ defines a rank function $\overline{\mathrm{rk}}_\Sigma$ on R_Σ/I such that, if $\pi : R_\Sigma\to R_\Sigma/I$ is the natural map, then $\mathrm{rk} = \lambda^\sharp(\mathrm{rk}_\Sigma) = \lambda^\sharp\pi^\sharp(\overline{\mathrm{rk}}_\Sigma)$. Therefore, $(R_\Sigma/I, \pi\circ\lambda)$ is another division envelope of rk, and hence isomorphic to $\mathcal{D}$ as R-ring, i.e., there exists an isomorphism $\psi : R_\Sigma/I\to\mathcal{D}$ such that the following commutes

$$\begin{tikzcd} & R \arrow[dl, "\pi\circ\lambda"'] \arrow[dr, "\phi"] & \\ R_\Sigma/I \arrow[rr, "\psi"] & & \mathcal{D} \end{tikzcd}$$

We deduce from this that
$$\tilde{\tau}\circ\psi\circ\pi\circ\lambda = \tilde{\tau}\circ\phi = \phi\circ\tau = \psi\circ\pi\circ\lambda\circ\tau = \psi\circ\pi\circ\bar{\tau}\circ\lambda,$$
what again by epicity of λ implies $\tilde{\tau}\circ(\psi\circ\pi) = (\psi\circ\pi)\circ\bar{\tau}$. Thus, the surjective map $\varphi = \psi\circ\pi : R_\Sigma\to\mathcal{D}$ extends to a surjective homomorphism
$$\tilde{\varphi} : R_\Sigma[t^{\pm1};\bar{\tau}]\to\mathcal{D}[t^{\pm};\tilde{\tau}]$$
such that $\tilde{\varphi}\circ\tilde{\lambda} = \tilde{\phi}$. We can check that since ψ is an isomorphism, the kernel of $\tilde{\varphi}$ is $\ker\tilde{\varphi} = \{\sum a_i t^i : a_i\in I \text{ for every } i\}$. In addition, we can see using Cramer's rule that rk_Σ is $\bar{\tau}$-compatible because rk is τ-compatible, and this means that if $a\in I$, then $\bar{\tau}^k(a)\in I$ for every integer k. Hence, $\ker\tilde{\varphi}$ is precisely the two-sided ideal generated by $j(I)$ in $R_\Sigma[t^{\pm1};\bar{\tau}]$. Since $\tilde{f}'\circ j = f$ and $\tilde{f}'$ is an isomorphism with inverse $\tilde{f}$, we have $\tilde{f}\circ f = j$, and hence the kernel of the composition
$$S_\Sigma \xrightarrow{\tilde{f}} R_\Sigma[t^{\pm1};\bar{\tau}] \xrightarrow{\tilde{\varphi}} \mathcal{D}[t^{\pm1};\tilde{\tau}]$$

is $\tilde{f}^{-1}(\ker \tilde{\varphi})$, i.e., the two-sided ideal J generated by $f(I)$ in S_Σ. Therefore, $S_\Sigma/J \cong \mathcal{D}[t^{\pm1};\tilde{\tau}]$ and this isomorphism is as S-rings because if $\pi' : S_\Sigma \to S_\Sigma/J$ is the natural map, the following commutes

$$\begin{tikzcd} & S \arrow[dl, "\pi'\circ\mu"'] \arrow[dr, "\tilde{\phi}=\tilde{\varphi}\circ\tilde{\lambda}"] & \\ S_\Sigma/J \arrow[rr, "\psi'", "\cong"'] & & \mathcal{D}[t^{\pm1};\tilde{\tau}]. \end{tikzcd}$$

Indeed, for every $s \in S$,

$$(\psi' \circ \pi' \circ \mu)(s) = \psi'(\mu(s) + J) = \tilde{\varphi}\tilde{f}(\mu(s)) = \tilde{\varphi}\tilde{\lambda}(s) = \tilde{\phi}(s).$$

Recall that rk' comes from the rank rk'_Σ on S_Σ via μ, and hence from the induced faithful rank on $S_\Sigma/\ker \mathrm{rk}'_\Sigma$. If we manage to show that $J \subseteq \ker \mathrm{rk}'_\Sigma$, then the commutativity of

$$\begin{tikzcd} & S_\Sigma \arrow[dl, "\pi'"'] \arrow[dr] & \\ S_\Sigma/J \arrow[rr] & & S_\Sigma/\ker \mathrm{rk}'_\Sigma \,. \end{tikzcd}$$

would imply that rk' comes from a rank rk'' on S_Σ/J via $\pi' \circ \mu$. But then $\mathrm{rk}_0 = ((\psi')^{-1})^\sharp(\mathrm{rk}'')$ defines a rank on $\mathcal{D}[t^{\pm1};\tilde{\tau}]$ and $\mathrm{rk}' = (\pi' \circ \mu)^\sharp(\mathrm{rk}'') = ((\psi')^{-1} \circ \tilde{\phi})^\sharp(\mathrm{rk}'') = \tilde{\phi}^\sharp(\mathrm{rk}_0)$, what finishes the proof as explained at the beginning.

Thus, it is left to show that $J \subseteq \ker \mathrm{rk}'_\Sigma$. Since the latter is a two-sided ideal, it suffices to show that $f(I) \subseteq \ker \mathrm{rk}'_\Sigma$. Take any $a \in I$ and use Cramer's rule to write $I_k \oplus a = P\lambda(A)Q$ for some non-negative integer k, a matrix A over R and P, Q invertible over R_Σ. As $f \circ \lambda = \mu \circ i$, the previous relation gives $I_k \oplus f(a) = P'(\mu \circ i)(A)Q'$, with P' and Q' invertible. Consequently, since rk' coincides with rk on R,

$$\mathrm{rk}'_\Sigma(f(a)) = \mathrm{rk}'_\Sigma(\mu \circ i(A)) - k = \mathrm{rk}'(i(A)) - k = \mathrm{rk}(A) - k = \mathrm{rk}_\Sigma(a) = 0,$$

as needed. □

5.2.2 Lück's approximation in the space of marked groups

In this subsection we introduce Lück's approximation conjecture in the space of marked groups and we prove it for virtually locally indicable groups.

A *k-marked group* is a group G together with a finite set $S = \{g_1, \dots, g_k\}$ of generators of G, and we say that two k-marked groups (G, S) and (G', S'), with $S' = \{g'_1, \dots, g'_k\}$, are *equivalent* if the correspondence $g_i \mapsto g'_i$ extends to a group isomorphism $G \to G'$. The set of k-marked groups up to this equivalence is usually denoted $\mathcal{G}_k$ and can be given the metric

$$d((G,S),(G',S')) = e^{-n},$$

where n is the largest integer such that $B_n(\mathrm{Cay}(G,S))$ and $B_n(\mathrm{Cay}(G',S'))$, which are respectively the balls of radius n centered at the neutral element in the Cayley graphs of G (with respect to S) and G' (with respect to S'), are isomorphic via an isomorphism respecting the labels.

Let F be a finitely k-generated free group with basis $X = \{x_1, \ldots, x_k\}$. The set $\mathcal{N}$ of normal subgroups of F can also be given a metric

$$d(N_1, N_2) = e^{-n},$$

where n is the largest integer such that

$$N_1 \cap B_n(\mathrm{Cay}(F, X)) = N_2 \cap B_n(\mathrm{Cay}(F, X)).$$

Observe that every k-marked group (G, S) induces a group homomorphism $\pi_S : F \to G$ by sending $x_i \mapsto g_i$, and that two k-marked groups (G, S) and (G', S') are equivalent if and only if there exists an isomorphism $G \cong G'$ such that the following commutes

$$\begin{tikzcd} & F \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_{S'}"] & \\ G \arrow[rr, "\cong"] & & G', \end{tikzcd}$$

i.e., if and only if they define the same normal subgroup $N = \ker \pi_S = \ker \pi_{S'}$ of F. Conversely, every normal subgroup N of F gives rise to a unique (up to equivalence) k-marked group $(F/N, \bar{X})$, where $\bar{X}$ is the image of X in the quotient. Thus, we have a bijection $\mathcal{N} \to \mathcal{G}_k$ that can be moreover seen to be a homeomorphism. In particular, N_i converges to N in $\mathcal{N}$ if and only if F/N_i converges to F/N in $\mathcal{G}_k$. In the sequel, we identify $\mathcal{N}$ and $\mathcal{G}_k$ without further comments, and following [Jai19], we use $\mathrm{MG}(F)$ to denote indistinctly any of these spaces.

If $\pi_G : F \to G$ is a surjective group homomorphism, we denote also by π_G the induced map $\pi_G : \mathbb{C}[F] \to \mathbb{C}[G]$. With this notation, Lück's approximation conjecture (over $\mathbb{C}$) in the space of marked groups can be stated as follows (cf. [Jai19, Conjecture 3]).

Conjecture (Lück's approximation conjecture over $\mathbb{C}$ in $\mathrm{MG}(F)$)**.** *Let F be a finitely generated free group, let $\{N_i\}_{i\in\mathbb{N}}$ converge to N in $\mathrm{MG}(F)$, and set $G_i = F/N_i$, $G = F/N$. Then, for every matrix A over $\mathbb{C}[F]$,*

$$\lim_{i\to\infty} \mathrm{rk}_{G_i}(\pi_{G_i}(A)) = \mathrm{rk}_G(\pi_G(A)).$$

In [Jai19, Corollary 1.4] it is proved that the conjecture holds if G_i is sofic for every i. We want to show that the conjecture also holds if the group G being approximated is virtually locally indicable.

Take a non-principal ultrafilter ω on $\mathbb{N}$. Since $\pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$ defines a rank function on $\mathbb{C}[F]$ for every i, we can define $\mathrm{rk}_{(\omega,G_i)} = \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$, the rank function on $\mathbb{C}[F]$ given by

$$\mathrm{rk}_{(\omega,G_i)}(A) = \lim_{\omega} \mathrm{rk}_{G_i}(\pi_{G_i}(A))$$

for every A over $\mathbb{C}[F]$ (see Corollary 1.4.15). If we manage to show that $\mathrm{rk}_{(\omega,G_i)} = \pi_G^\sharp(\mathrm{rk}_G)$ as rank functions on $\mathbb{C}[F]$ for every non-principal ultrafilter ω on $\mathbb{N}$, then we would have that, for every matrix A over $\mathbb{C}[F]$ and every such ω,

$$\lim_\omega \mathrm{rk}_{G_i}(\pi_{G_i}(A)) = \mathrm{rk}_G(\pi_G(A)),$$

what means by Proposition 1.4.13 that the actual limit exists and equals $\mathrm{rk}_G(\pi_G(A))$, as we want to show. Thus, what we are actually going to show is the following equivalent form of the conjecture for a virtually locally indicable group G.

Conjecture (Lück's approximation conjecture over $\mathbb{C}$ in $\mathrm{MG}(F)$)**.** *Let F be a finitely generated free group, let $\{N_i\}_{i\in\mathbb{N}}$ converge to N in $\mathrm{MG}(F)$, and set $G_i = F/N_i$, $G = F/N$. Then, for every non-principal ultrafilter ω on $\mathbb{N}$,*

$$\lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i}) = \pi_G^\sharp(\mathrm{rk}_G).$$

From Kazhdan's inequality (see [Jai19S, Proposition 10.7]), and taking into account on the one hand the relation between $\dim_{\mathcal{N}(G)} \ker r_{A,\ell^2}$ and $\mathrm{rk}_G(A)$ given in Section 4.2, and on the other hand the relation between ultralimits and limits of convergent subsequences, we can already deduce one inequality.

Proposition 5.2.7. *Let F be a finitely generated free group, let $\{N_i\}_{i\in\mathbb{N}}$ converge to N in $\mathrm{MG}(F)$, and set $G_i = F/N_i$, $G = F/N$. For every non-principal ultrafilter ω on $\mathbb{N}$, we have $\pi_G^\sharp(\mathrm{rk}_G) \leq \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$.*

From now on, we fix a non-principal ultrafilter ω on $\mathbb{N}$. If F is a finitely generated free group and $G = F/N$ for some normal subgroup $N \triangleleft F$, then the associated homomorphism $\pi_G : \mathbb{C}[F] \to \mathbb{C}[G]$ is actually a $*$-homomorphism, and one can show that its kernel is the two-sided ideal I of $\mathbb{C}[F]$ generated by the set $\{x - 1 : x \in N\}$. Moreover, since in $\mathbb{C}[F]$ we have $(x-1)^* = x^{-1} - 1$, the previous set, and hence I, is $*$-closed. Thus, the quotient $\mathbb{C}[F]/I$ is a $*$-ring with the involution $(a+I)^* = a^* + I$, and hence we have a $*$-isomorphism $\mathbb{C}[F]/I \to \mathbb{C}[G]$ defined by $a + I \mapsto \pi_G(a)$. This will be used in the following lemma, where we record some facts about $\lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$.

Lemma 5.2.8. *Let F be a finitely generated free group, let $\{N_i\}_{i\in\mathbb{N}}$ converge to N in $\mathrm{MG}(F)$, and set $G_i = F/N_i$, $G = F/N$. Then*

1. *The Sylvester matrix rank function $\lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$ on $\mathbb{C}[F]$ is $*$-regular with positive definite $*$-regular envelope.*

2. *There exists a faithful $*$-regular Sylvester matrix rank function rk on $\mathbb{C}[G]$ with positive definite $*$-regular envelope such that*

$$\lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i}) = \pi_G^\sharp(\mathrm{rk}).$$

Proof.
1. For each j, the induced map $\pi_{G_j} : \mathbb{C}[F] \to \mathbb{C}[G_j]$ and the inclusion map $\iota_j : \mathbb{C}[G_j] \to \mathcal{R}_{\mathbb{C}[G_j]}$ are $*$-homomorphisms, and hence $(\mathcal{R}_{\mathbb{C}[G_j]}, \mathrm{rk}_{G_j}, \iota_j \circ \pi_{G_j})$ is the $*$-regular (positive definite) envelope of $\pi^{\sharp}_{G_j}(\mathrm{rk}_{G_j})$.

Set $\mathcal{R} = \prod_j \mathcal{R}_{\mathbb{C}[G_j]}$, which is $*$-regular (with the component-wise involution) and positive definite, and let $p_{G_j} : \mathcal{R} \to \mathcal{R}_{\mathbb{C}[G_j]}$ denote the canonical map. For the fixed non-principal ultrafilter ω on $\mathbb{N}$, we can construct the rank function $\mathrm{rk}_\omega = \lim_\omega p^{\sharp}_{G_j}(\mathrm{rk}_{G_j})$ on $\mathcal{R}$, and note that if we consider the map $\pi = (\pi_{G_j}) : \mathbb{C}[F] \to \prod_j \mathbb{C}[G_j]$ and the inclusion $\iota : \prod_j \mathbb{C}[G_j] \to \mathcal{R}$, which are both $*$-homomorphisms, then

$$\lim_{\omega} \pi^{\sharp}_{G_j}(\mathrm{rk}_{G_j}) = (\iota \circ \pi)^{\sharp}(\mathrm{rk}_\omega)$$

Indeed,

$$\begin{aligned}(\iota \circ \pi)^{\sharp}(\mathrm{rk}_\omega) &= \lim\nolimits_\omega (\iota \circ \pi)^{\sharp} p^{\sharp}_{G_j}(\mathrm{rk}_{G_j}) = \lim\nolimits_\omega (p_{G_j} \circ \iota \circ \pi)^{\sharp}(\mathrm{rk}_{G_j}) \\ &= \lim\nolimits_\omega (\iota_j \circ \pi_{G_j})^{\sharp}(\mathrm{rk}_{G_j}) = \lim\nolimits_\omega (\pi_{G_j})^{\sharp}(\mathrm{rk}_{G_j}),\end{aligned}$$

where the last equality follows because $\iota^{\sharp}_j(\mathrm{rk}_{G_j})$ is precisely rk_{G_j} as a rank on $\mathbb{C}[G_j]$. Now, by Lemma 1.3.11 $\ker \mathrm{rk}_\omega$ is a two-sided ideal of $\mathcal{R}$ and rk_ω induces a faithful rank function rk'_ω on $\mathcal{R}_\omega := \mathcal{R}/\ker \mathrm{rk}_\omega$ so that, if we consider the canonical map $p_\omega : \mathcal{R} \to \mathcal{R}_\omega$, then $\mathrm{rk}_\omega = p^{\sharp}_\omega(\mathrm{rk}'_\omega)$. Furthermore, Corollary 4.1.4 tells us that $\mathcal{R}_\omega$ is $*$-regular with the involution $(a + \ker \mathrm{rk}_\omega)^* = a^* + \ker \mathrm{rk}_\omega$, what makes p_ω a $*$-homomorphism.

Moreover, we claim that $\mathcal{R}_\omega$ is positive definite. Indeed, $\mathrm{Mat}_n(\mathcal{R})$ and $\mathrm{Mat}_n(\mathcal{R}_\omega)$ are $*$-rings with the $*$-transpose involution, and the former is $*$-regular because $\mathcal{R}$ is positive definite (see Lemma 4.1.25). The induced map $p^{(n)}_\omega : \mathrm{Mat}_n(\mathcal{R}) \to \mathrm{Mat}_n(\mathcal{R}_\omega)$ is a surjective $*$-homomorphism, and hence, again by Corollary 4.1.4 and the first isomorphism theorem, $\mathrm{Mat}_n(\mathcal{R}_\omega)$ is $*$-isomorphic to the $*$-regular ring $\mathrm{Mat}_n(\mathcal{R})/\ker p^{(n)}_\omega$, and hence $*$-regular. Since this is valid for every n, Lemma 4.1.25 tells us that $\mathcal{R}_\omega$ is positive definite.

Adding everything up, the composition $\pi_\omega := p_\omega \circ \iota \circ \pi$ defines a $*$-homomorphism $\pi_\omega : \mathbb{C}[F] \to \mathcal{R}_\omega$ such that $\pi^{\sharp}_\omega(\mathrm{rk}'_\omega) = \lim_\omega \pi^{\sharp}_{G_j}(\mathrm{rk}_{G_j})$. Therefore, if we set $\mathcal{U} = \mathcal{R}(\pi_\omega(\mathbb{C}[F]), \mathcal{R}_\omega)$, the $*$-regular closure of the image of $\mathbb{C}[F]$ in $\mathcal{R}_\omega$, then $(\mathcal{U}, \mathrm{rk}'_\omega, \pi_\omega)$ is the $*$-regular positive definite envelope of $\lim_\omega \pi^{\sharp}_{G_j}(\mathrm{rk}_{G_j})$, what finishes the proof of 1.

2. To prove 2. we are going to study the kernel of the map $\pi_\omega : \mathbb{C}[F] \to \mathcal{R}_\omega$. More precisely, we claim that $\ker \pi_\omega = \ker \pi_G =: I$, the two-sided ideal generated by the set $\{x - 1 : x \in N\}$.

On the one hand, if $x \in N$, then since N_j converges to N, there exists n such that for every $j \geq n$, $\pi_{G_j}(x-1) = 0$ in $\mathbb{C}[G_j]$. Thus, $\mathrm{rk}_{G_j}(\pi_{G_j}(x-1)) = 0$ for every $j \geq n$, and consequently

$$\mathrm{rk}_\omega((\iota \circ \pi)(x-1)) = \lim_{\omega} \mathrm{rk}_{G_j}(\pi_{G_j}(x-1)) = \lim_{j\to\infty} \mathrm{rk}_{G_j}(\pi_{G_j}(x-1)) = 0.$$

Hence, $\iota \circ \pi(x-1) \in \ker \mathrm{rk}_\omega = \ker p_\omega$, i.e., $x - 1 \in \ker \pi_\omega$. Since the latter is a two-sided ideal, we deduce that $I \subseteq \ker \pi_\omega$.

On the other hand, Proposition 5.2.7 gives the inequality $\pi_G^\sharp(\mathrm{rk}_G) \leq \lim_\omega \pi_{G_j}^\sharp(\mathrm{rk}_{G_j}) = \pi_\omega^\sharp(\mathrm{rk}'_\omega)$. Therefore, if $a \in \ker \pi_\omega$,

$$\mathrm{rk}_G(\pi_G(a)) \leq \mathrm{rk}'_\omega(\pi_\omega(a)) = \mathrm{rk}'_\omega(0) = 0,$$

what implies by faithfulness of rk_G that $\pi_G(a) = 0$, i.e., $a \in \ker \pi_G = I$. This gives the other containment $\ker \pi_\omega \subseteq I$, and hence $\ker \pi_\omega = I$.

We already discussed that I is $*$-closed, and hence since π_ω is a $*$-homomomorphism, we obtain a composition of $*$-isomorphisms

$$\mathbb{C}[G] \cong \mathbb{C}[F]/I = \mathbb{C}[F]/\ker \pi_\omega \cong \pi_\omega(\mathbb{C}[F])$$

By definition, if $a \in \mathbb{C}[G]$ is of the form $\pi_G(b)$ for some $b \in \mathbb{C}[F]$, then via this composition $a \mapsto b + I \mapsto \pi_\omega(b)$. Therefore, we have a well-defined commutative diagram

$$\begin{tikzcd} & \mathbb{C}[F] \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_\omega"] & \\ \mathbb{C}[G] \arrow[rr, "\phi"] & & \mathcal{R}_\omega, \end{tikzcd}$$

where ϕ is given by the previous composition, i.e., $\phi(a) = \pi_\omega(b)$ if $a = \pi_G(b)$. Observing that ϕ is injective, $\mathrm{rk} = \phi^\sharp(\mathrm{rk}'_\omega)$ is a faithful rank function on $\mathbb{C}[G]$ such that

$$\pi_G^\sharp(\mathrm{rk}) = \pi_G^\sharp \phi^\sharp(\mathrm{rk}'_\omega) = \pi_\omega^\sharp(\mathrm{rk}'_\omega) = \lim_\omega \pi_{G_j}^\sharp(\mathrm{rk}_{G_j}),$$

as we wanted to show. Finally, ϕ is a $*$-homomorphism with image $\pi_\omega(\mathbb{C}[F])$, what means that $\mathcal{R}(\phi(\mathbb{C}[G]), \mathcal{R}_\omega) = \mathcal{R}(\pi_\omega(\mathbb{C}[F]), \mathcal{R}_\omega) = \mathcal{U}$ (constructed in 1.), and therefore $(\mathcal{U}, \mathrm{rk}'_\omega, \phi)$ is the $*$-regular positive definite envelope of rk. This finishes the proof of the lemma. □

We shall also need for our purposes to understand the relation between natural extensions of a rank function on $\mathbb{C}[G]$ and on $\mathbb{C}[F]$. This has been extracted from the proof of [JL20, Propositions 7.8 & 7.9].

Lemma 5.2.9. *Let F be a finitely generated free group, let $M \triangleleft F$ be a normal subgroup of F and set $G = F/M$. If $N \triangleleft G$ is a normal subgroup of G with $G/N \cong \mathbb{Z}$, then:*

(i) $N' = \pi_G^{-1}(N)$ is a normal subgroup of F containing M satisfying $F/N' \cong \mathbb{Z}$. Moreover, $\pi_G : \mathbb{C}[F] \to \mathbb{C}[G]$ restricts to a surjective map $\pi_N : \mathbb{C}[N'] \to \mathbb{C}[N]$, and if $x \in F$ satisfies $F = \langle N', x \rangle$, then $G = \langle N, \bar{x} \rangle$, with $\bar{x} = \pi_G(x)$.

(ii) We have a commutative diagram

$$\begin{tikzcd} \mathbb{C}[F] \arrow[r, "\psi'", "\cong"'] \arrow[d, "\pi_G"'] & \mathbb{C}[N'][t^{\pm 1}; \tau_x] \arrow[d, "\widetilde{\pi}_N"] \\ \mathbb{C}[G] \arrow[r, "\cong", "\psi"'] & \mathbb{C}[N][t^{\pm 1}; \tau_{\bar{x}}] \end{tikzcd}$$

Here, ψ and ψ' are the usual isomorphisms, with τ_x and $\tau_{\bar{x}}$ given by left conjugation by x and $\bar{x}$, respectively, and $\widetilde{\pi}_N$ acts as π_N on $\mathbb{C}[N']$ and sends $t \mapsto t$.

(iii) Let rk_1 and rk_2 be $*$-regular Sylvester matrix rank functions on $\mathbb{C}[G]$, and consider $\mathrm{rk}_1' = \pi_G^\sharp(\mathrm{rk}_1)$, $\mathrm{rk}_2' = \pi_G^\sharp(\mathrm{rk}_2)$. If we have

$$\mathrm{rk}_1' = (\psi')^\sharp\left(\widetilde{\mathrm{rk}'_{2|\mathbb{C}[N']}}\right)$$

then

$$\mathrm{rk}_1 = \psi^\sharp\left(\widetilde{\mathrm{rk}_{2|\mathbb{C}[N]}}\right)$$

In other words, if rk_1' is the natural extension of the restriction of rk_2' to $\mathbb{C}[N']$, then rk_1 is the natural extension of the restriction of rk_2 to $\mathbb{C}[N]$.

Proof.
(i) Observe that $N' = \pi_G^{-1}(N)$ is the kernel of the composition of group homomorphisms $F \xrightarrow{\pi_G} G \to G/N$. Hence, it contains M and it is a normal subgroup of F with $F/N' \cong G/N \cong \mathbb{Z}$. Since π_G is surjective, $\pi_G(N') = N$ and therefore we have, on the one hand, that π_G restricts to a surjective homomorphism $\pi_N : \mathbb{C}[N'] \to \mathbb{C}[N]$ and, on the other hand, that if $F = \langle N', x\rangle$, then $G = \pi_G(F) = \langle N, \bar{x}\rangle$.

(ii) Now, since $F/N' = \langle N'x\rangle$ and $G/N = \langle N\bar{x}\rangle$, we know that we can define the isomorphisms ψ and ψ'. For this, recall that we can write $\mathbb{C}[F] = \bigoplus_i \mathbb{C}[N']x^i$ (resp. $\mathbb{C}[G] = \bigoplus_i \mathbb{C}[N]\bar{x}^i$) and ψ' is the isomorphism acting as the identity on $\mathbb{C}[N']$ and sending $x \mapsto t$ (resp. ψ is the isomorphism acting as the identity on $\mathbb{C}[N]$ and sending $\bar{x} \mapsto t$).

In addition, since π_N is the restriction of π_G, for every $a \in \mathbb{C}[N']$,

$$(\pi_N \circ \tau_x)(a) = \pi_N(xax^{-1}) = \pi_G(x)\pi_N(a)\pi_G(x)^{-1} = (\tau_{\bar{x}} \circ \pi_N)(a).$$

This implies that $\pi_N \circ \tau_x = \tau_{\bar{x}} \circ \pi_N$, and therefore the map $\widetilde{\pi}_N$ given in (ii) is a well-defined homomorphism making the diagram commute.

(iii) Set $\mathrm{rk}' = \mathrm{rk}'_{2|\mathbb{C}[N']}$ and $\mathrm{rk} = \mathrm{rk}_{2|\mathbb{C}[N]}$. Since $x \in F$ and $\bar{x} \in G$, the automorphisms τ_x and $\tau_{\bar{x}}$ are actually $*$-isomorphisms. In addition, as rk' and rk are restrictions of rank functions on $\mathbb{C}[F]$ and $\mathbb{C}[G]$, respectively, and $x, \bar{x}$ are units, we obtain that rk' and rk are compatible with the corresponding $*$-automorphisms. Observe also that $\mathrm{rk}' = \pi_N^\sharp(\mathrm{rk})$.

Moreover, rk is $*$-regular because rk_2 is $*$-regular. Let $(\mathcal{U}, \mathrm{rk}_0, \phi)$ be the $*$-regular envelope of rk. Then, by Proposition 4.1.28, there exists a $*$-automorphism τ of $\mathcal{U}$ such that $\tau \circ \phi = \phi \circ \tau_{\bar{x}}$ and rk_0 is τ-compatible, and we can extend ϕ to a homomorphism $\tilde{\phi} : \mathbb{C}[N][t^{\pm 1}; \tau_{\bar{x}}] \to \mathcal{U}[t^{\pm 1}; \tau]$ sending $t \mapsto t$ in such a way that

$$\widetilde{\mathrm{rk}} = \tilde{\phi}^\sharp(\widetilde{\mathrm{rk}}_0).$$

Taking into account that π_N is a surjective $*$-homomorphism and that $\mathrm{rk}' = \pi_N^\sharp(\mathrm{rk})$, we have that $(\mathcal{U}, \mathrm{rk}_0, \phi \circ \pi_N)$ is the $*$-regular envelope of rk'. In addition, $\tau \circ (\phi \circ$

$\pi_N) = \phi \circ \tau_{\bar{x}} \circ \pi_N = (\phi \circ \pi_N) \circ \tau_x$, we can extend $\varphi = \phi \circ \pi_N$ to a homomorphism $\tilde{\varphi} : \mathbb{C}[N'][t^{\pm 1}; \tau_x] \to \mathcal{U}[t^{\pm 1}; \tau]$ sending $t \mapsto t$, and following the same proposition we obtain that $\widetilde{\mathrm{rk}}' = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}_0)$. Noticing that $\tilde{\varphi} = \tilde{\phi} \circ \widetilde{\pi}_N$, we deduce in particular that

$$\widetilde{\mathrm{rk}}' = \tilde{\varphi}^\sharp(\widetilde{\mathrm{rk}}_0) = \widetilde{\pi}_N^\sharp \tilde{\phi}^\sharp(\widetilde{\mathrm{rk}}_0) = \widetilde{\pi}_N^\sharp(\widetilde{\mathrm{rk}}).$$

Consequently, if we have

$$\mathrm{rk}_1' = (\psi')^\sharp(\widetilde{\mathrm{rk}}')$$

then by the commutativity of the diagram in (ii),

$$\begin{aligned} \pi_G^\sharp(\mathrm{rk}_1) &= \mathrm{rk}_1' = (\psi')^\sharp(\widetilde{\mathrm{rk}}') = (\psi')^\sharp \widetilde{\pi}_N^\sharp(\widetilde{\mathrm{rk}}) = (\widetilde{\pi}_N \circ \psi')^\sharp(\widetilde{\mathrm{rk}}) \\ &= (\psi \circ \pi_G)^\sharp(\widetilde{\mathrm{rk}}) = \pi_G^\sharp \psi^\sharp(\widetilde{\mathrm{rk}}). \end{aligned}$$

The surjectivity of π_G finally implies that $\mathrm{rk}_1 = \psi^\sharp(\widetilde{\mathrm{rk}})$, which is precisely what we wanted to show. □

At some point during the main proposition used to prove Lück's approximation conjecture for a virtually locally indicable group G, we shall change the approximation in $\mathrm{MG}(F)$ and we shall need then to compare between different ultralimits of ranks.

More precisely, assume that M_i converges to M in $\mathrm{MG}(F)$, set $G = F/M$ and assume that there exists a normal subgroup $N \triangleleft G$ with $G/N \cong \mathbb{Z}$. By Lemma 5.2.9(i), $N' = \pi_G^{-1}(N)$ is normal in F, $M \leq N'$ and $F/N' \cong \mathbb{Z}$. Then $K_i = M_i \cap N'$ also converges to M in $\mathrm{MG}(F)$, and therefore, setting $G_i = F/M_i$ and $H_i = F/K_i$, we can consider the following three Sylvester matrix rank functions on $\mathbb{C}[F]$:

$$\mathrm{rk}_1' = \pi_G^\sharp(\mathrm{rk}_G) \quad \mathrm{rk}_2' = \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i}) \quad \mathrm{rk}_3' = \lim_\omega \pi_{H_i}^\sharp(\mathrm{rk}_{H_i}).$$

In addition, Lemma 5.2.8 2. tells us that there exist faithful rank functions rk_2 and rk_3 on $\mathbb{C}[G]$ such that $\mathrm{rk}_2' = \pi_G^\sharp(\mathrm{rk}_2)$ and $\mathrm{rk}_3' = \pi_G^\sharp(\mathrm{rk}_3)$. These Sylvester matrix rank functions are related as follows.

Proposition 5.2.10. *In the above setting,*

(1) rk_3' *is the natural extension of the restriction of* rk_2' *to* $\mathbb{C}[N']$*. Hence,* rk_3 *is the natural extension of the restriction of* rk_2 *to* $\mathbb{C}[N]$*.*

(2) $\mathrm{rk}_1' \leq \mathrm{rk}_2' \leq \mathrm{rk}_3'$ *as rank functions on* $\mathbb{C}[F]$*. Hence,* $\mathrm{rk}_G \leq \mathrm{rk}_2 \leq \mathrm{rk}_3$ *as rank functions on* $\mathbb{C}[G]$*.*

Proof.
(1) Since $K_i \leq N'$, we have that $N_i = N'/K_i$ is a normal subgroup of H_i satisfying by the third isomorphism theorem that $H_i/N_i \cong F/N' \cong \mathbb{Z}$. Moreover, by definition, we have $N' = \pi_{H_i}^{-1}(N_i)$, and hence Lemma 5.2.9(i) tells us that $\pi_{H_i} : \mathbb{C}[F] \to \mathbb{C}[H_i]$ restricts to a surjective homomorphism $\pi_{N_i} : \mathbb{C}[N'] \to \mathbb{C}[N_i]$ and that, if $x \in F$ satisfies $F = \langle N', x\rangle$, then we have $H_i = \langle N_i, \bar{x}_i\rangle$ with $\bar{x}_i = \pi_{H_i}(x)$.

Now, Lemma 5.2.9(ii) gives the commutativity of the following diagram for every i.

$$\begin{tikzcd} \mathbb{C}[F] \arrow[r, "\psi'", "\cong"'] \arrow[d, "\pi_{H_i}"'] & \mathbb{C}[N'][t^{\pm 1}; \tau_x] \arrow[d, "\widetilde{\pi}_{N_i}"] \\ \mathbb{C}[H_i] \arrow[r, "\cong", "\psi_i"'] & \mathbb{C}[N_i][t^{\pm 1}; \tau_{\bar{x}_i}] \end{tikzcd}$$

Since rk_{N_i} is the restriction of rk_{H_i} to $\mathbb{C}[N_i]$ by Proposition 4.2.2, one can prove as in the proof of Lemma 5.2.9(iii) that $\widetilde{\pi}^\sharp_{N_i}(\widetilde{\mathrm{rk}}_{N_i})$ is the natural extension of $\pi^\sharp_{N_i}(\mathrm{rk}_{N_i})$. Moreover, each of the rank functions $\pi^\sharp_{N_i}(\mathrm{rk}_{N_i})$ is $*$-regular as a rank on $\mathbb{C}[N']$ because rk_{N_i} is $*$-regular and π_{N_i} a $*$-homomorphism, and they are all compatible with the $*$-automorphism τ_x because rk_{N_i} is $\tau_{\bar{x}_i}$-compatible and $\bar{x}_i = \pi_{H_i}(x)$. Thus, we can apply Corollary 4.1.29, that tells us that, as a rank function on $\mathbb{C}[N'][t^{\pm 1}; \tau_x]$,

$$\lim_\omega \widetilde{\pi}^\sharp_{N_i}(\widetilde{\mathrm{rk}}_{N_i}) \text{ is the natural extension of } \lim_\omega \pi^\sharp_{N_i}(\mathrm{rk}_{N_i}) \tag{5.3}$$

Observe also that, since $H_i/N_i \cong \mathbb{Z}$, Proposition 4.2.7 says that rk_{H_i}, as a rank function on $\mathbb{C}[H_i]$, is the natural extension of rk_{N_i}, meaning that

$$\mathrm{rk}_{H_i} = \psi_i^\sharp(\widetilde{\mathrm{rk}}_{N_i}) \tag{5.4}$$

Finally, since $K_i \leq M_i$, we have a surjective homomorphism $H_i \to G_i$. The composition $\varphi : N_i \to H_i \to G_i$ is then injective, since φ sends $K_i a \mapsto M_i a$ for every $a \in N'$, and hence $\varphi(K_i a) = 1_{G_i}$ if and only if $a \in M_i$. In this event, $a \in N' \cap M_i = K_i$ and therefore $K_i a = 1_{N_i}$. The discussion at the beginning of the section shows that $\mathrm{rk}_{N_i} = \varphi^\sharp(\mathrm{rk}_{G_i})$, and the commutativity of

$$\begin{tikzcd} \mathbb{C}[N'] \arrow[r, "\iota"] \arrow[d, "\pi_{N_i}"'] & \mathbb{C}[F] \arrow[d, "\pi_{G_i}"] \\ \mathbb{C}[N_i] \arrow[r, "\varphi"'] & \mathbb{C}[G_i], \end{tikzcd}$$

where ι denotes the inclusion map, shows that

$$\begin{aligned} \pi^\sharp_{N_i}(\mathrm{rk}_{N_i}) &= \pi^\sharp_{N_i}\varphi^\sharp(\mathrm{rk}_{G_i}) = (\varphi \circ \pi_{N_i})^\sharp(\mathrm{rk}_{G_i}) \\ &= (\pi_{G_i} \circ \iota)^\sharp(\mathrm{rk}_{G_i}) = \iota^\sharp(\pi^\sharp_{G_i}(\mathrm{rk}_{G_i})) \end{aligned}$$

and hence

$$\left.\begin{aligned} \lim_\omega \pi^\sharp_{N_i}(\mathrm{rk}_{N_i}) &= \lim_\omega \iota^\sharp(\pi^\sharp_{G_i}(\mathrm{rk}_{G_i})) = \iota^\sharp\left(\lim_\omega \pi^\sharp_{G_i}(\mathrm{rk}_{G_i})\right) \\ &= \iota^\sharp(\mathrm{rk}'_2) = \mathrm{rk}'_{2|\mathbb{C}[N']} \end{aligned}\right. \tag{5.5}$$

Adding equations (5.3), (5.4) and (5.5), we obtain that

$$\begin{aligned} \mathrm{rk}'_3 &= \lim_\omega \pi^\sharp_{H_i}(\mathrm{rk}_{H_i}) \overset{(5.4)}{=} \lim_\omega \pi^\sharp_{H_i}\psi_i^\sharp(\widetilde{\mathrm{rk}}_{N_i}) = \lim_\omega (\psi_i \circ \pi_{H_i})^\sharp(\widetilde{\mathrm{rk}}_{N_i}) \\ &= \lim_\omega (\psi_i \circ \pi_{H_i})^\sharp(\widetilde{\mathrm{rk}}_{N_i}) = \lim_\omega (\widetilde{\pi}_{N_i} \circ \psi')^\sharp(\widetilde{\mathrm{rk}}_{N_i}) = \lim_\omega (\psi')^\sharp \widetilde{\pi}^\sharp_{N_i}(\widetilde{\mathrm{rk}}_{N_i}) \\ &= (\psi')^\sharp\left(\lim_\omega \widetilde{\pi}^\sharp_{N_i}(\widetilde{\mathrm{rk}}_{N_i})\right) \overset{(5.3),(5.5)}{=} (\psi')^\sharp\left(\widetilde{\mathrm{rk}'_{2|\mathbb{C}[N']}}\right) \end{aligned}$$

which is exactly what we wanted to prove.

By choice of N and N', Lemma 5.2.9(ii) also gives a commutative diagram

$$\begin{tikzcd} \mathbb{C}[F] \arrow[r, "\psi'", "\cong"'] \arrow[d, "\pi_G"'] & \mathbb{C}[N'][t^{\pm 1}; \tau_x] \arrow[d, "\widetilde{\pi}_N"] \\ \mathbb{C}[G] \arrow[r, "\cong", "\psi"'] & \mathbb{C}[N][t^{\pm 1}; \tau_{\bar{x}}] \end{tikzcd}$$

with $\bar{x} = \pi_G(x)$, and since rk_2 and rk_3 are $*$-regular on $\mathbb{C}[G]$ satisfying $\mathrm{rk}_2' = \pi_G^\sharp(\mathrm{rk}_2)$ and $\mathrm{rk}_3' = \pi_G^\sharp(\mathrm{rk}_3)$ by Lemma 5.2.8(2), then Lemma 5.2.9(iii) states precisely that $\mathrm{rk}_3 = \psi^\sharp\left(\widetilde{\mathrm{rk}_{2|\mathbb{C}[N]}}\right)$, what finishes the proof of (1).

(2) The inequalities $\mathrm{rk}_1' \leq \mathrm{rk}_2'$ and $\mathrm{rk}_1' \leq \mathrm{rk}_3'$ are consequences of Proposition 5.2.7, since we are working with two approximations of the same normal subgroup of F. It is left to see that $\mathrm{rk}_2' \leq \mathrm{rk}_3'$.

For each i, set $G_i' = G_i \times F/N'$, and consider the natural homomorphisms $\pi_1 : F \to G_i'$ and $\pi_2 : G_i' \to G_i$ with $\pi_1(f) = (M_i f, N' f)$ and $\pi_2(M_i f, N' f) \mapsto M_i f$, for every $f \in F$. Let π_1 and π_2 also denote the induced homomorphisms of group rings and observe that $\pi_2 \circ \pi_1 = \pi_{G_i}$.

Now, observe that $f \in F$ belongs to $\ker \pi_1$ if and only if $f \in M_i \cap N' = K_i$, so that $H_i = F/\ker \pi_1 \cong \pi_1(F) \leq G_i'$. Thus, we have an embedding $\phi : H_i \to G_i'$, what implies by the discussion at the beginning of the section that $\phi^\sharp(\mathrm{rk}_{G_i'}) = \mathrm{rk}_{H_i}$. Noticing that $\phi \circ \pi_{H_i} = \pi_1$ we obtain that

$$\pi_{H_i}^\sharp(\mathrm{rk}_{H_i}) = \pi_{H_i}^\sharp \phi^\sharp(\mathrm{rk}_{G_i'}) = (\phi \circ \pi_{H_i})^\sharp(\mathrm{rk}_{G_i'}) = \pi_1^\sharp(\mathrm{rk}_{G_i'}).$$

Moreover, as $F/N' \cong \mathbb{Z}$, we have by Corollary 5.2.5 that $\pi_2^\sharp(\mathrm{rk}_{G_i}) \leq \mathrm{rk}_{G_i'}$, and hence

$$\pi_{G_i}^\sharp(\mathrm{rk}_{G_i}) = \pi_1^\sharp \pi_2^\sharp(\mathrm{rk}_{G_i}) \leq \pi_1^\sharp(\mathrm{rk}_{G_i'}) = \pi_{H_i}^\sharp(\mathrm{rk}_{H_i}).$$

Since this is valid for every i, we have by Lemma 1.4.12 that $\mathrm{rk}_2' \leq \mathrm{rk}_3'$, as we wanted to show. The last assertion of the proposition follows because we have proved that $\pi_G^\sharp(\mathrm{rk}_G) \leq \pi_G^\sharp(\mathrm{rk}_2) \leq \pi_G^\sharp(\mathrm{rk}_3)$ and π_G is surjective. $\square$

Let us make a final observation in the form of a lemma before passing to the main results. Let F be a finitely generated free group, let M_i converge to M in $\mathrm{MG}(F)$ and set $G = F/M$, $G_i = F/M_i$. If H is a non-trivial finitely generated subgroup of G, then there exists a non-trivial finitely generated subgroup F' of F whose image under $F' \hookrightarrow F \to G$ is H, i.e., $H = F'M/M$.

Consider $H_i = F'M_i/M_i$, and write $M_i' = F' \cap M_i$, $M' = F' \cap M$. Then $H \cong F'/M' =: H'$ and $H_i \cong F'/M_i' =: H_i'$ by the second isomorphism theorem, and note that M_i' converges to M' in $\mathrm{MG}(F')$. Let $\pi_H, \pi_{H'}, \pi_{H_i}$ and $\pi_{H_i'}$ denote the homomorphisms from F' (resp. $\mathbb{C}[F']$) to the corresponding group (resp. group ring). Then we have the following.

Lemma 5.2.11. *In the previous setting,*

(i) As Sylvester matrix rank functions on $\mathbb{C}[F']$,

$$\left(\lim_{\omega} \pi^{\sharp}_{G_i}(\mathrm{rk}_{G_i})\right)_{|\mathbb{C}[F']} = \lim_{\omega} \pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}).$$

(ii) If rk and rk' are the rank functions on $\mathbb{C}[G]$ and $\mathbb{C}[H']$ satisfying

$$\pi^{\sharp}_{G}(\mathrm{rk}) = \lim_{\omega} \pi^{\sharp}_{G_i}(\mathrm{rk}_{G_i}) \qquad \pi^{\sharp}_{H'}(\mathrm{rk}') = \lim_{\omega} \pi^{\sharp}_{H_i'}(\mathrm{rk}_{H_i'}),$$

and if φ denotes the isomorphism $H \to H'$, then $\mathrm{rk}_{|\mathbb{C}[H]} = \varphi^{\sharp}(\mathrm{rk}')$, and in particular

$$\pi^{\sharp}_{H}(\mathrm{rk}_{|\mathbb{C}[H]}) = \lim_{\omega} \pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}).$$

Proof. Let φ_i denote the isomorphism $H_i \to H_i'$, so that by the remark at the beginning of the section we have $\varphi_i^{\sharp}(\mathrm{rk}_{H_i'}) = \mathrm{rk}_{H_i}$. Noting that $\varphi_i \circ \pi_{H_i} = \pi_{H_i'}$ we deduce that as Sylvester matrix rank functions on $\mathbb{C}[F']$,

$$\lim_{\omega} \pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}) = \lim_{\omega} \pi^{\sharp}_{H_i}\varphi_i^{\sharp}(\mathrm{rk}_{H_i'}) = \lim_{\omega} \pi^{\sharp}_{H_i'}(\mathrm{rk}_{H_i'}).$$

The existence of rk and rk' as in (ii) is guaranteed by Lemma 5.2.8, since M_i converges to M in $\mathrm{MG}(F)$ and M_i' converges to M' in $\mathrm{MG}(F')$. The previous relation, together with the fact that $\varphi \circ \pi_H = \pi_{H'}$, also shows that the rank $\varphi^{\sharp}(\mathrm{rk}')$ on $\mathbb{C}[H]$ satisfies

$$\pi^{\sharp}_{H}(\varphi^{\sharp}(\mathrm{rk}')) = \pi^{\sharp}_{H'}(\mathrm{rk}') = \lim_{\omega} \pi^{\sharp}_{H_i'}(\mathrm{rk}_{H_i'}) = \lim_{\omega} \pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}).$$

Now, the commutativity of the diagram of group rings

$$\begin{tikzcd} \mathbb{C}[F'] \arrow[r, "\pi_{H_i}"] \arrow[d, "\iota"'] & \mathbb{C}[H_i] \arrow[d, "j_i"] \\ \mathbb{C}[F] \arrow[r, "\pi_{G_i}"'] & \mathbb{C}[G_i] \end{tikzcd}$$

where ι and j_i are inclusion maps, shows that $\pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}) = \pi^{\sharp}_{H_i} j_i^{\sharp}(\mathrm{rk}_{G_i}) = \iota^{\sharp}(\pi^{\sharp}_{G_i}(\mathrm{rk}_{G_i}))$ and hence that, as rank functions on $\mathbb{C}[F']$,

$$\lim_{\omega} \pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}) = \lim_{\omega} \iota^{\sharp}(\pi^{\sharp}_{G_i}(\mathrm{rk}_{G_i})) = \iota^{\sharp}\left(\lim_{\omega} \pi^{\sharp}_{G_i}(\mathrm{rk}_{G_i})\right)$$

which is precisely the statement of (i). From the corresponding diagram for $j : \mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$ we can finally read

$$\begin{aligned} \pi^{\sharp}_{H}(\mathrm{rk}_{|\mathbb{C}[H]}) &= \pi^{\sharp}_{H} j^{\sharp}(\mathrm{rk}) = \iota^{\sharp}(\pi^{\sharp}_{G}(\mathrm{rk})) = \iota^{\sharp}\left(\lim_{\omega} \pi^{\sharp}_{G_i}(\mathrm{rk}_{G_i})\right) \\ &= \lim_{\omega} \pi^{\sharp}_{H_i}(\mathrm{rk}_{H_i}) = \pi^{\sharp}_{H}(\varphi^{\sharp}(\mathrm{rk}')) \end{aligned}$$

and the surjectivity of π_H implies that $\mathrm{rk}_{|\mathbb{C}[H]} = \varphi^{\sharp}(\mathrm{rk}')$, finishing the proof. $\square$

Since φ is an isomorphism of groups, the associated ring homomorphism $\varphi : \mathbb{C}[H] \to \mathbb{C}[H']$ is a $*$-isomorphism, and hence, if the $*$-regular and faithful rank function rk' appearing in (ii) has $*$-regular positive definite envelope $(\mathcal{U}, \mathrm{rk}_0, \phi)$ (see Lemma 5.2.8 2.), then $\varphi^\sharp(\mathrm{rk}')$ is also faithful and $*$-regular on $\mathbb{C}[H]$ with $*$-regular positive definite envelope $(\mathcal{U}, \mathrm{rk}_0, \phi \circ \varphi)$. In view of this and the previous lemma, in the following result we make an abuse of language and say that the convergence M_i' to M' in $\mathrm{MG}(F')$ is associated to an approximation of H (instead of H').

Observe also that if F is a finitely generated free group, every convergent sequence $M_i \to M$ in $\mathrm{MG}(F)$ has a unique group $G = F/M$ and a unique $*$-regular rank function $\lim_\omega \pi^\sharp_{F/M_i}(\mathrm{rk}_{F/M_i})$ on $\mathbb{C}[F]$ associated to it, and hence defines an epic $*$-regular (positive definite) $\mathbb{C}[F]$-ring, namely the $*$-regular envelope of the previous rank, which is also unique (up to isomorphism of epic $*$-regular $\mathbb{C}[F]$-rings) by Corollary 4.1.21.

Proposition 5.2.12. *Let F be a finitely generated free group, let M_i converge to M in $\mathrm{MG}(F)$, set $G_i = F/M_i$, $G = F/M$, and assume that G is locally indicable. Let rk be the faithful $*$-regular Sylvester matrix rank function on $\mathbb{C}[G]$ satisfying*

$$\pi^\sharp_G(\mathrm{rk}) = \lim_\omega \pi^\sharp_{G_i}(\mathrm{rk}_{G_i}),$$

and let $(\mathcal{U}, \mathrm{rk}'_\omega, \phi)$ denote its positive definite $*$-regular envelope. If $\varphi = (\iota, \phi)$ denotes the map $\mathbb{C}[G] \to \mathcal{R}_{\mathbb{C}[G]} \times \mathcal{U}$, then the division closure of $\varphi(\mathbb{C}[G])$ in $\mathcal{R}_{\mathbb{C}[G]} \times \mathcal{U}$ is a division ring.

Proof. Set $\mathcal{S} = \mathcal{R}_{\mathbb{C}[G]} \times \mathcal{U}$, which by the preceding discussion is uniquely determined by the approximation, and let $\mathcal{D}_{G,\mathcal{S}}$ denote the division closure of $\varphi(\mathbb{C}[G])$ in $\mathcal{S}$. In addition, for every subgroup $H \leq G$ set $\mathcal{S}_H = \mathcal{R}_{\mathbb{C}[H]} \times \mathcal{U}_H$, where $\mathcal{U}_H$ denotes the $*$-regular closure of $\phi(\mathbb{C}[H])$ in $\mathcal{U}$, and let $\mathcal{D}_{H,\mathcal{S}}$ denote the division closure of $\varphi(\mathbb{C}[H])$ in $\mathcal{S}$. Note that $\varphi(\mathbb{C}[H]) \subseteq \mathcal{S}_H$ and that the latter is regular, so that $\mathcal{D}_{H,\mathcal{S}} = \mathcal{D}_{H,\mathcal{S}_H}$ by Lemma 3.3.3.

We are going to simultaneously prove the result for every finitely generated locally indicable group G of the form $G = F/M$, where F is a finitely generated free group, and for all approximations M_i of M in $\mathrm{MG}(F)$, by using induction on the complexity. More precisely, we are going to show that if G is any finitely generated locally indicable group, M_i is any approximation of M in $\mathrm{MG}(F)$ with $F/M = G$ and $\alpha \in \mathrm{Rat}(\mathbb{C}^\times G)$ realizes the G-complexity of a non-zero element $a \in \mathcal{D}_{G,\mathcal{S}}$, then a is invertible in $\mathcal{D}_{G,\mathcal{S}}$. Since the morphism of rational $\mathbb{C}^\times G$-semirings $\Phi_{G,\mathcal{S}} : \mathrm{Rat}(\mathbb{C}^\times G) \to \mathcal{D}_{G,\mathcal{S}}$ is surjective for every choice of G and $\mathcal{S}$ (i.e., of approximation) by Proposition 4.3.9, this gives the result.

First, observe that for every finitely generated locally indicable group G and for every approximation M_i of M in $\mathrm{MG}(F)$ such that $F/M = G$, if $\alpha \in \mathrm{Rat}(\mathbb{C}^\times G)$ satisfies $\mathrm{Tree}(\alpha) = 1_{\mathcal{T}}$ and realizes the G-complexity of a non-zero element $a \in \mathcal{D}_{G,\mathcal{S}}$, then $\alpha \in \mathbb{C}^\times G$ by Lemma 4.3.7(ii) and hence $a = \Phi_{G,\mathcal{S}}(\alpha) = \varphi(\alpha) \in \varphi(\mathbb{C}^\times G)$ is invertible.

Now assume that we have a finitely generated locally indicable group G, an approximation M_i of M in $\mathrm{MG}(F)$ with $F/M = G$ and an element $\alpha \in \mathrm{Rat}(\mathbb{C}^\times G)$ with $\mathrm{Tree}(\alpha) > 1_{\mathcal{T}}$ realizing the G-complexity of a non-zero element $a \in \mathcal{D}_{G,\mathcal{S}}$, and that we have already proved that for every finitely generated locally indicable group G', for every approximation M_i' of M' in $\mathrm{MG}(F')$ with $F'/M' = G'$ and for every $\beta \in \mathrm{Rat}(\mathbb{C}^\times G')$

with $\mathrm{Tree}(\beta) < \mathrm{Tree}(\alpha)$ realizing the G'-complexity of non-zero element $a' \in \mathcal{D}_{G',\mathcal{S}'}$, a' is invertible in $\mathcal{D}_{G',\mathcal{S}'}$.

As usual, we can assume that α is primitive (see the proof of Theorem 4.3.14), and hence if H is the image of $\mathrm{source}(\alpha)$ via $\mathbb{C}^\times G \to \mathbb{C}^\times G/\mathbb{C}^\times \cong G$, then H is finitely generated, $\alpha \in \mathrm{Rat}(\mathbb{C}^\times H)$ and $a \in \mathcal{D}_{H,\mathcal{S}} = \mathcal{D}_{H,\mathcal{S}_H}$ (see Proposition 4.3.9). In particular, α realizes the H-complexity of a and H, as a subgroup of G, is locally indicable.

If H is trivial, then $\mathbb{C}[H] = \mathbb{C}$ and hence $\varphi(\mathbb{C}[H]) \cong \mathbb{C}$ is division closed and $a \in \mathcal{D}_{H,\mathcal{S}_H} = \varphi(\mathbb{C}[H]) \cong \mathbb{C}$ is invertible. Otherwise, let F' be a finitely generated subgroup of F such that H is the image of F' under the composition $F' \hookrightarrow F \to G$, i.e., $H = F'M/M \cong F'/F' \cap M = H'$. Denote by δ the isomorphism $H \cong H'$ and as usual let π_H, $\pi_{H'}$ be the corresponding homomorphisms $F' \to H$ and $F' \to H'$ and their associated maps of group rings. Since H is non-trivial, there exists a normal subgroup $N \lhd H$ such that $H/N \cong \mathbb{Z}$, and hence since δ is an isomorphism, $\delta(N)$ is normal in H' with $H'/\delta(N) \cong \mathbb{Z}$. By Lemma 5.2.9, $N' = \pi_{H'}^{-1}(\delta(N)) = \pi_H^{-1}(N)$ is a normal subgroup of F' containing $F' \cap M$ and such that $F'/N' \cong \mathbb{Z}$.

This implies that we have two different approximations of $M' = F' \cap M$ in $\mathrm{MG}(F')$. On the one hand, if we set $M_i' = F' \cap M_i$, then as discussed before Lemma 5.2.11, M_i' converges to M'. On the other hand, setting $K_i = M_i' \cap N' = F' \cap M_i \cap N'$, we also have that K_i converges to M' in $\mathrm{MG}(F')$ (because $F' \cap M \subseteq N'$). Set $H_i = F'M_i/M_i$, $H_i' = F'/M_i' \cong H_i$ and $H_i'' = F'/K_i$, and consider the following three Sylvester matrix rank functions on $\mathbb{C}[F]$

$$\mathrm{rk}_1' = \pi_{H'}^\sharp(\mathrm{rk}_{H'}), \quad \mathrm{rk}_2' = \lim_\omega \pi_{H_i'}^\sharp(\mathrm{rk}_{H_i'}), \quad \mathrm{rk}_3' = \lim_\omega \pi_{H_i''}^\sharp(\mathrm{rk}_{H_i''}).$$

Now, Lemma 5.2.8 2. tells us that there exist faithful $*$-regular rank functions rk_2'' and rk_3'' on $\mathbb{C}[H']$ such that $\pi_{H'}^\sharp(\mathrm{rk}_2'') = \mathrm{rk}_2'$ and $\pi_{H'}^\sharp(\mathrm{rk}_3'') = \mathrm{rk}_3'$, and Proposition 5.2.10 then shows that

1. rk_3'' is the natural extension of the restriction of rk_2'' to $\mathbb{C}[\delta(N)]$.

2. $\mathrm{rk}_{H'} \leq \mathrm{rk}_2'' \leq \mathrm{rk}_3''$ as rank function on $\mathbb{C}[H']$.

Since δ, as a map $\mathbb{C}[H] \to \mathbb{C}[H']$, is a $*$-isomorphism, we have that $\delta^\sharp(\mathrm{rk}_{H'}) = \mathrm{rk}_H$, and if we set $\mathrm{rk}_2 = \delta^\sharp(\mathrm{rk}_2'')$ and $\mathrm{rk}_3 = \delta^\sharp(\mathrm{rk}_3'')$, we have that rk_2 and rk_3 are faithful and $*$-regular and the previous relations reveal that

1. rk_3 is the natural extension of the restriction of rk_2 to $\mathbb{C}[N]$.

2. $\mathrm{rk}_H \leq \mathrm{rk}_2 \leq \mathrm{rk}_3$ as rank function on $\mathbb{C}[H]$.

Let us identify their $*$-regular envelopes. In the first place, the $*$-regular envelope of rk_H is $\mathcal{R}_{\mathbb{C}[H]}$, and observe that by Lemma 5.2.11(ii) we have $\mathrm{rk}_2 = \delta^\sharp(\mathrm{rk}_2'') = \mathrm{rk}_{|\mathbb{C}[H]}$. Therefore, the $*$-regular envelope of rk_2 is precisely $(\mathcal{U}_H, \mathrm{rk}_\omega', \phi)$. Finally, to identify the $*$-regular envelope of rk_3 we use the fact that it is the natural extension of $\mathrm{rk}_{2|\mathbb{C}[N]} = \mathrm{rk}_{|\mathbb{C}[N]}$.

If $x \in H$ is such that $H/N = \langle Nx \rangle$ and we consider the $*$-isomorphism $\psi : \mathbb{C}[H] \to \mathbb{C}[N][t^{\pm 1}; \tau_x]$ acting as the identity on $\mathbb{C}[N]$ and taking $x \mapsto t$, where τ_x is the $*$-automorphism of $\mathbb{C}[N]$ induced by left conjugation by x, then the last assertion means

that
$$\mathrm{rk}_3 = \psi^\sharp\left(\widetilde{\mathrm{rk}_{|\mathbb{C}[N]}}\right)$$
Since $\mathrm{rk}_{|\mathbb{C}[N]}$ is a $*$-regular and τ_x-compatible (because it is the restriction of a rank on $\mathbb{C}[G]$ and x is invertible) rank function with positive definite (because $\mathcal{U}$ is positive definite) $*$-regular envelope $(\mathcal{U}_N, \mathrm{rk}'_\omega, \phi)$, we can apply Proposition 4.1.28 (see also Proposition 4.1.26, Remark 4.1.27). This proposition tells us that there exist a $*$-automorphism τ_2 of $\mathcal{U}_N$ such that $\tau_2 \circ \phi = \phi \circ \tau_x$ (here, we see $\phi : \mathbb{C}[N] \to \mathcal{U}_N$), a $*$-map $\tilde{\phi} : \mathbb{C}[N][t^{\pm 1}; \tau_x] \to \mathcal{U}_N[t^{\pm 1}; \tau_2]$ that extends ϕ and maps $t \mapsto t$, a $*$-regular ring $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tau_2}$ (for the fixed non-principal ultrafilter ω on $\mathbb{N}$) with a faithful rank function rk_0 and an injective $*$-homomorphism $f_\omega : \mathcal{U}_N[t^{\pm 1}; \tau_2] \to \mathcal{P}^{\mathcal{U}_N}_{\omega,\tau_2}$ such that
$$\widetilde{\mathrm{rk}_{|\mathbb{C}[N]}} = (f_\omega \circ \tilde{\phi})^\sharp(\mathrm{rk}_0).$$
Therefore,
$$\mathrm{rk}_3 = \psi^\sharp\left(\widetilde{\mathrm{rk}_{|\mathbb{C}[N]}}\right) = \psi^\sharp(f_\omega \circ \tilde{\phi})^\sharp(\mathrm{rk}_0) = (f_\omega \circ \tilde{\phi} \circ \psi)^\sharp(\mathrm{rk}_0),$$
i.e., if we set $\mathcal{U}' = \mathcal{R}(f_\omega \circ \tilde{\phi} \circ \psi(\mathbb{C}[H]), \mathcal{P}^{\mathcal{U}_N}_{\omega,\tau_2})$, then $(\mathcal{U}', \mathrm{rk}_0, f_\omega \circ \tilde{\phi} \circ \psi)$ is the $*$-regular envelope of rk_3, and it can be shown to be positive definite. Indeed, rk_3 admits a $*$-regular positive definite envelope by Lemma 5.2.8 2., and hence by uniqueness of the $*$-regular envelope we have in particular a $*$-isomorphism between this one and $\mathcal{U}'$, proving the claim.

If we denote $\mathcal{B} = \mathcal{R}_{\mathbb{C}[H]} \times \mathcal{U}'$ and $\mathcal{K} = \mathcal{R}_{\mathbb{C}[H]} \times \mathcal{U}_H \times \mathcal{U}'$, then since $\mathrm{rk}_H \le \mathrm{rk}_2 \le \mathrm{rk}_3$ as rank functions on $\mathbb{C}[H]$ and we have identified their $*$-regular envelopes, Corollary 5.2.3 (and its proof) tells us that the following diagram commutes
$$\begin{tikzcd} & \mathrm{Rat}(\mathbb{C}^\times H) \arrow[dl, "\Phi_{H,\mathcal{S}_H}"'] \arrow[d, "\Phi_{\mathcal{K}}"'] \arrow[dr, "\Phi_{H,\mathcal{B}}"] & \\ \mathcal{D}_{H,\mathcal{S}_H} & \mathcal{D}_{H,\mathcal{K}} \arrow[l, "\pi_{12}"'] \arrow[r, "\pi_{13}"] & \mathcal{D}_{H,\mathcal{B}}, \end{tikzcd}$$
that $a' = \Phi_{H,\mathcal{B}}(\alpha)$ is non-zero (since $a = \Phi_{H,\mathcal{S}_H}(\alpha)$ is non-zero) and that a is invertible if and only if a' is invertible. Here, if $\varphi' = (\iota, f_\omega \circ \tilde{\phi} \circ \psi)$, then $\mathcal{D}_{H,\mathcal{B}}$ denotes the division closure of $\varphi'(\mathbb{C}[H])$ in $\mathcal{B}$.

Observe that, by definition, $\mathrm{Tree}_H(a') \le \mathrm{Tree}(\alpha)$, and recall that $H \cong H' = F'/M'$ is a finitely generated locally indicable group and that K_i converges to M' in $\mathrm{MG}(F')$. Moreover, recall that $H''_i = F'/K_i$ and that rk_3 is a faithful $*$-regular rank function on $\mathbb{C}[H]$ such that
$$\pi^\sharp_H(\mathrm{rk}_3) = \pi^\sharp_H \delta^\sharp(\mathrm{rk}''_3) = \pi^\sharp_{H'}(\mathrm{rk}''_3) = \mathrm{rk}'_3 = \lim_\omega \pi^\sharp_{H''_i}(\mathrm{rk}_{H''_i})$$
and with positive definite $*$-regular envelope $(\mathcal{U}', \mathrm{rk}_0, f_\omega \circ \tilde{\phi} \circ \psi)$. Thus, by the induction hypothesis we have that every non-zero $b \in \mathcal{D}_{H,\mathcal{B}}$ with $\mathrm{Tree}_H(b) < \mathrm{Tree}(\alpha)$ is invertible. In particular, if $\mathrm{Tree}_H(a') < \mathrm{Tree}(\alpha)$, we obtain that a', and hence a by the previous reasoning, is invertible.

It is left to study the case in which $\mathrm{Tree}_H(a') = \mathrm{Tree}(\alpha)$, i.e., in which α realizes the H-complexity of a'. For this case, we want to apply as usual Proposition 4.3.13, and for that we need conditions (i), (ii) and (iii) of that proposition to be satisfied.

(i) Consider $\mathcal{A} = \mathcal{S}_N = \mathcal{R}_{\mathbb{C}[N]} \times \mathcal{U}_N$, which is a regular ring, together with the restriction $\varphi : \mathbb{C}[N] \to \mathcal{A}$.

(ii) Since $\mathcal{R}_{\mathbb{C}[N]}$ is the $*$-regular envelope of rk_N, Proposition 4.1.28 tells us that there exists a $*$-automorphism τ_1 of $\mathcal{R}_{\mathbb{C}[N]}$ such that $\tau_1 \circ \iota = \iota \circ \tau_x$, where ι denotes here the embedding $\mathbb{C}[N] \hookrightarrow \mathcal{R}_{\mathbb{C}[N]}$, and that ι extends to a homomorphism $\tilde{\iota} : \mathbb{C}[N][t^{\pm 1}; \tau_x] \to \mathcal{R}_{\mathbb{C}[N]}[t^{\pm 1}; \tau_1]$ sending $t \mapsto t$. Thus, $\tau = (\tau_1, \tau_2)$ is an automorphism of $\mathcal{A}$ such that $\tau \circ \varphi = \varphi \circ \tau_x$.

(iii) As in Eq. (4.3), f_ω can actually be extended to an embedding $f_\omega : \mathcal{U}_N((t; \tau_2)) \to \mathcal{P}^{\mathcal{U}_N}_{\omega,\tau_2}$, and hence, if we let Δ denote the isomorphism $\mathcal{A}((t;\tau)) \cong \mathcal{R}_{\mathbb{C}[N]}((t;\tau_1)) \times \mathcal{U}_N((t;\tau_2))$, then we will take

$$\mathcal{P} = \mathcal{R}_{\mathbb{C}[N]}((t;\tau_1)) \times \mathcal{P}^{\mathcal{U}_N}_{\omega,\tau_2}$$

together with the embedding $(\mathrm{id}, f_\omega) \circ \Delta : \mathcal{A}((t;\tau)) \to \mathcal{P}$.

Recall from Theorem 4.4.2 that $\mathcal{R}_{\mathbb{C}[G]}$ is the Hughes-free division $\mathbb{C}[G]$-ring of fractions, and hence we have an isomorphism of $\mathbb{C}[H]$-rings $\mathcal{R}_{\mathbb{C}[H]} \cong \mathcal{R}_{\mathbb{C}[N]}(t;\tau_1)$ (see the proof of Proposition 3.4.31), what gives a natural embedding $j : \mathcal{R}_{\mathbb{C}[H]} \to \mathcal{R}_{\mathbb{C}[N]}((t;\tau_1))$. In addition, if $j' : \mathcal{U}' \to \mathcal{P}^{\mathcal{U}_N}_{\omega,\tau_2}$ is the inclusion map one can carefully show that the following diagram commutes

$$\begin{tikzcd}
\mathbb{C}[N] \arrow[r] \arrow[d, "\varphi"'] & \mathbb{C}[H] \arrow[r, "\varphi'"] \arrow[d, "\tilde{\varphi}"] & \mathcal{B} \arrow[d, "{(j,j')}"] \\
\mathcal{A} \arrow[r] & \mathcal{A}((t;\tau)) \arrow[r, "f"'] & \mathcal{P},
\end{tikzcd}$$

where $\tilde{\varphi}$ is the map acting as φ on $\mathbb{C}[N]$ and sending $x \mapsto t$.

As $\mathcal{B}$ is regular and (j, j') is injective, Lemma 3.3.3 gives $\mathcal{D}_{H,\mathcal{B}} \cong \mathcal{D}_{H,(j,j')(\mathcal{B})} = \mathcal{D}_{H,\mathcal{P}}$ as $\mathbb{C}[H]$-rings, where $\mathcal{D}_{H,\mathcal{P}}$ denotes the division closure of $f\tilde{\varphi}(\mathbb{C}[H])$ in $\mathcal{P}$. Consequently, Lemma 4.3.11 tells us that

$$\begin{tikzcd}
& \mathrm{Rat}(\mathbb{C}^{\times} H) \arrow[dl, "\Phi_{H,\mathcal{B}}"'] \arrow[dr, "\Phi_{H,\mathcal{P}}"] & \\
\mathcal{D}_{H,\mathcal{B}} \arrow[rr, "\cong"'] & & \mathcal{D}_{H,\mathcal{P}},
\end{tikzcd}$$

commutes and that the H-complexity of an element in $\mathcal{D}_{H,\mathcal{B}}$ coincides with the H-complexity of its image in $\mathcal{D}_{H,\mathcal{P}}$ because it is realized by the same element of $\mathrm{Rat}(\mathbb{C}^{\times} H)$. In particular, if a'' is the image of a' in $\mathcal{D}_{H,\mathcal{P}}$, then α realizes its H-complexity, and since every non-zero $b \in \mathcal{D}_{H,\mathcal{B}}$ with $\mathrm{Tree}_H(b) < \mathrm{Tree}(\alpha) = \mathrm{Tree}_H(a')$ was invertible by the induction hypothesis, the same holds for any non-zero element in $\mathcal{D}_{H,\mathcal{P}}$ strictly less H-complex than a''.

Thus, Proposition 4.3.13 applies and tells us that $a'' = f(\bar{c})$ for some $\bar{c} = \sum_k c_k \in \mathcal{D}_{N,\mathcal{A}}((t;\tau))$ with $c_k \in \mathcal{D}_{N,\mathcal{A}}t^k$ and $\mathrm{Tree}_H(f(c_k)) \leq \mathrm{Tree}_H(a'')$. Moreover, we claim that there are at least two non-zero summands. Otherwise, if $\bar{c} = c_n$, then $a'' = f(c_n)$ and $\mathrm{Tree}_H(a'') = \mathrm{Tree}_H(f(c_n))$, from where the same proposition tells us that $\alpha \in \mathrm{Rat}(\mathbb{C}^\times N)x^n \subseteq \mathrm{Rat}(\mathbb{C}^\times N)\mathbb{C}^\times G$. This would imply by Theorem 4.3.8(iv) that $\mathrm{source}(\alpha) \leq \mathbb{C}^\times N$, and hence that $H \leq N$, a contradiction.

Hence, $\mathrm{Tree}_H(f(c_k)) < \mathrm{Tree}_H(a'')$ for all k. In particular this is true for n, the smallest k such that c_k is non-zero. Assume that $c_n = yt^n$ for some $y \in \mathcal{D}_{N,\mathcal{A}}$. Remark 4.3.12(2.) tells us that f restricts to an isomorphism $\mathcal{D}_{N,\mathcal{A}} \cong \mathcal{D}_{N,\mathcal{P}}$, and therefore $f(y) \in \mathcal{D}_{N,\mathcal{P}} \subseteq \mathcal{D}_{H,\mathcal{P}}$. Since $t^n = \tilde{\varphi}(x^n)$, we also have $f(t^n) = f\tilde{\varphi}(x^n) \in f\tilde{\varphi}(\mathbb{C}[H]) \subseteq \mathcal{D}_{H,\mathcal{P}}$. As a consequence $f(c_n)$ is an element in $\mathcal{D}_{H,\mathcal{P}}$ with strictly less H-complexity than a'', so as before we have that $f(c_n)$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$. Since t is invertible, this implies that $f(y)$ is invertible in $\mathcal{D}_{H,\mathcal{P}}$, and hence in $\mathcal{D}_{N,\mathcal{P}}$. Again, since f is an isomorphism from $\mathcal{D}_{N,\mathcal{A}}$ to $\mathcal{D}_{N,\mathcal{P}}$, we conclude that y is invertible in $\mathcal{D}_{N,\mathcal{A}}$. This implies that $\bar{c}$ is invertible in $\mathcal{D}_{N,\mathcal{A}}((t;\tau))$, and therefore $f(\bar{c}) = a''$ is invertible in $\mathcal{P}$, and hence in $\mathcal{D}_{H,\mathcal{P}}$ since it is division closed. Therefore, a' is invertible in $\mathcal{D}_{H,\mathcal{B}}$, and this implies that a is invertible in $\mathcal{D}_{H,\mathcal{S}_H} \subseteq \mathcal{D}_{G,\mathcal{S}}$, as we wanted to show. □

We can now prove Lück's approximation conjecture for virtually locally indicable groups.

Theorem 5.2.13. *Let F be a finitely generated free group, let $\{M_i\}_{i\in\mathbb{N}}$ converge to M in $\mathrm{MG}(F)$, and set $G_i = F/M_i$, $G = F/M$. If G is virtually locally indicable, then for every non-principal ultrafilter ω on $\mathbb{N}$,*

$$\lim_{\omega} \pi_{G_i}^{\sharp}(\mathrm{rk}_{G_i}) = \pi_G^{\sharp}(\mathrm{rk}_G).$$

Proof. We consider three different cases throughout the proof.

Case 1: G is locally indicable.

Let us fix a non-principal ultrafilter ω on $\mathbb{N}$, let rk be the faithful $*$-regular Sylvester matrix rank function on $\mathbb{C}[G]$ such that $\pi_G^{\sharp}(\mathrm{rk}) = \lim_\omega \pi_{G_i}^{\sharp}(\mathrm{rk}_{G_i})$ and let $(\mathcal{U}, \mathrm{rk}'_\omega, \phi)$ denote its $*$-regular positive definite envelope (as in Lemma 5.2.8 2.). If we set $\mathcal{S} = \mathcal{R}_{\mathbb{C}[G]} \times \mathcal{U}$, $\varphi = (\iota, \phi) : \mathbb{C}[G] \to \mathcal{S}$, then Proposition 5.2.12 tells us that the division closure $\mathcal{D}_{G,\mathcal{S}}$ of $\varphi(\mathbb{C}[G])$ in $\mathcal{S}$ is a division ring. If $\pi_1 : \mathcal{D}_{G,\mathcal{S}} \to \mathcal{R}_{\mathbb{C}[G]}$ and $\pi_2 : \mathcal{D}_{G,\mathcal{S}} \to \mathcal{U}$ denote the restrictions of the canonical projections from $\mathcal{S}$ to each factor, then since $\mathcal{D}_{G,\mathcal{S}}$ is a division ring, we have that they are both injective. Moreover, as the following diagrams commute

$$\begin{tikzcd} \mathbb{C}[G] \arrow[r, "\varphi"] \arrow[d, "\iota"'] & \mathcal{D}_{G,\mathcal{S}} \arrow[d, "\pi_1"] \\ \mathcal{R}_{\mathbb{C}[G]} \arrow[r, "\mathrm{id}"'] & \mathcal{R}_{\mathbb{C}[G]} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}[G] \arrow[r, "\varphi"] \arrow[d, "\phi"'] & \mathcal{D}_{G,\mathcal{S}} \arrow[d, "\pi_2"] \\ \mathcal{U} \arrow[r, "\mathrm{id}"'] & \mathcal{U} \end{tikzcd}$$

with φ, ι and ϕ epic, and $\mathcal{D}_{G,\mathcal{S}}, \mathcal{R}_{\mathbb{C}[G]}$ and $\mathcal{U}$ regular rings, Corollary 4.1.15 tells us that $\pi_1(\mathcal{D}_{G,\mathcal{S}}) = \mathcal{R}_{\mathbb{C}[G]}$ and $\pi_2(\mathcal{D}_{G,\mathcal{S}}) = \mathcal{U}$. Therefore, π_1 and π_2 are also surjective, and hence isomorphisms.

From here, $\delta = \pi_2 \circ \pi_1^{-1}$ defines an isomorphism from $\mathcal{R}_{\mathbb{C}[G]}$ to $\mathcal{U}$ making the following commute

$$\begin{tikzcd} & \mathbb{C}[G] \arrow[dl, "\iota"'] \arrow[dr, "\phi"] & \\ \mathcal{R}_{\mathbb{C}[G]} \arrow[rr, "\delta"] & & \mathcal{U} \end{tikzcd}$$

In particular, since $\mathcal{R}_{\mathbb{C}[G]}$ is a division ring and rk_G its unique rank function, we have that $\mathrm{rk}_G = \delta^\sharp(\mathrm{rk}'_\omega)$, and consequently as rank functions on $\mathbb{C}[G]$,

$$\mathrm{rk} = \phi^\sharp(\mathrm{rk}'_\omega) = \iota^\sharp(\delta^\sharp(\mathrm{rk}'_\omega)) = \iota^\sharp(\mathrm{rk}_G) = \mathrm{rk}_G\,.$$

Thus, $\pi_G^\sharp(\mathrm{rk}_G) = \pi_G^\sharp(\mathrm{rk}) = \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$, and since this is valid for every non-principal ultrafilter ω on $\mathbb{N}$, this finishes the proof.

Case 2: G is virtually locally indicable and ICC.

Let us fix a non-principal ultrafilter ω on $\mathbb{N}$. In this case, there exists a finite index locally indicable subgroup G' of G. Let us show first that G' can be assumed to be normal and of the form F'/M for some finitely generated normal subgroup F' of F.

Indeed, if G' is a finite index locally indicable subgroup of G, then $F' = \pi_G^{-1}(G')$ is a subgroup of F containing M and with $|F : F'| = |G : G'| < \infty$. Since F' has finite index, it has a finite number of conjugate subgroups, and the intersection F'' of all of them is on the one hand normal in F, and on the other hand finite index as the intersection of a finite number of finite index subgroups. Additionally, it contains M since M is normal and hence a subgroup of each conjugate subgroup of F', and it is finitely generated and free by Nielsen-Schreier's formula. Since π_G is surjective, $G'' = \pi_G(F'') = F''/M$ is a finitely generated normal subgroup of G, with finite index $|G : G''| = |F/M : F''/M| = |F : F''| < \infty$ and locally indicable since it is a subgroup of G'. This proves the claim.

Let rk be the faithful $*$-regular Sylvester matrix rank function on $\mathbb{C}[G]$ such that $\pi_G^\sharp(\mathrm{rk}) = \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$ and let $(\mathcal{U}, \mathrm{rk}'_\omega, \phi)$ denote its $*$-regular positive definite envelope (as in Lemma 5.2.8 2.). Now, M_i converges to M in $\mathrm{MG}(F)$, and if we set $M' = F' \cap M$ and $M'_i = F' \cap M_i$, we have that M'_i converges to M' in $\mathrm{MG}(F')$. Setting $G'_i = F'/M'_i$, Lemma 5.2.11 tells us that if $\pi_{G'}$ and $\pi_{G'_i}$ are the induced maps from $\mathbb{C}[F']$ to $\mathbb{C}[G']$ and $\mathbb{C}[G'_i]$, respectively, then the faithful $*$-regular rank function rk' on $\mathbb{C}[G']$ satisfying $\pi_{G'}^\sharp(\mathrm{rk}') = \lim_\omega \pi_{G'_i}^\sharp(\mathrm{rk}_{G'_i})$ is precisely $\mathrm{rk}' = \mathrm{rk}_{|\mathbb{C}[G']}$, and the previous case asserts in addition that $\mathrm{rk}' = \mathrm{rk}_{G'}$.

Thus, if $\mathcal{U}' = \mathcal{R}(\phi(\mathbb{C}[G']), \mathcal{U})$ is the $*$-regular closure of $\phi(\mathbb{C}[G'])$ in $\mathcal{U}$, we have that both $(\mathcal{U}', \mathrm{rk}'_\omega, \phi)$ and $\mathcal{R}_{\mathbb{C}[G']}$ are $*$-regular envelopes of rk'. By uniqueness of the $*$-regular envelope (Corollary 4.1.21), there exists in particular a $*$-isomorphism δ' such that the following commute

$$\begin{tikzcd} & \mathbb{C}[G'] \arrow[dl, "\iota"'] \arrow[dr, "\phi"] & \\ \mathcal{R}_{\mathbb{C}[G']} \arrow[rr, "\delta'"] & & \mathcal{U}' \end{tikzcd}$$

and $\mathrm{rk}_{G'} = \delta'^{\sharp}(\mathrm{rk}'_\omega)$. We are going to show that δ' extends to a $*$-isomorphism between $\mathcal{R}_{\mathbb{C}[G]}$ and $\mathcal{U}$.

Given the facts that $G' \triangleleft G$, G/G' is finite and $\mathcal{R}_{\mathbb{C}[G']} = \mathcal{D}_{\mathbb{C}[G']}$ is a division ring, [Lin98, Lemmas 9.2, 9.3 & 9.4] state that $\mathcal{D}_{\mathbb{C}[G]}$, the division closure of $\mathbb{C}[G]$ in $\mathcal{U}(G)$, is semisimple artinian and $*$-closed, hence $*$-regular, and therefore $\mathcal{D}_{\mathbb{C}[G]} = \mathcal{R}_{\mathbb{C}[G]}$. Furthermore, they tell us that if $\{x_1, \dots, x_n\}$ is a transversal of G' in G, then x_i normalizes $\mathcal{R}_{\mathbb{C}[G']}$ for every i and we can write $\mathcal{R}_{\mathbb{C}[G]} = \mathcal{R}_{\mathbb{C}[G']}G = \bigoplus x_i\mathcal{R}_{\mathbb{C}[G']}$, where $\mathcal{R}_{\mathbb{C}[G']}G$ denotes the subring generated by $\mathcal{R}_{\mathbb{C}[G']}$ and G.

Now, since G' satisfies the strong Atiyah conjecture over $\mathbb{C}$ and has finite index, we have by [Lin98, Lemma 8.6] (see also [Sch00**, Proposition 2.1]) that for every matrix A over $\mathbb{C}[G]$, $\mathrm{rk}_G(A) \in \frac{1}{|G:G'|}\mathbb{Z}$. From Proposition 4.1.17 and the faithfulness of rk_G we obtain in particular that if $a \neq b$ are elements in $\mathcal{R}_{\mathbb{C}[G]}$, then $\mathrm{rk}_G(a-b) \geq \frac{1}{|G:G'|}$, and as a consequence, $\mathcal{R}_{\mathbb{C}[G]}$ coincides with its rk_G-completion. Therefore, since G is ICC, [Jai19, Propositions 5.8 & 5.7] gives that $\mathcal{R}_{\mathbb{C}[G]}$ is actually simple with a unique Sylvester matrix rank function rk_G.

We define the map $\delta : \mathcal{R}_{\mathbb{C}[G]} \to \mathcal{U}$ by $\delta(x_i a) = \phi(x_i)\delta'(a)$ for $a \in \mathcal{R}_{\mathbb{C}[G']}$. The previous expression of $\mathcal{R}_{\mathbb{C}[G]}$ as a direct sum shows that this defines a well-defined linear map. One can check that, in order to prove that it defines a ring homomorphism, it is enough to show the equality $\phi(x_i)^{-1}\delta'(a)\phi(x_i) = \delta'(x_i^{-1}ax_i)$ over $\mathcal{U}$ for every i and $a \in \mathcal{R}_{\mathbb{C}[G']}$. This is equivalent to proving that $\delta' \circ \tau_{(x_i)^{-1}} = \tau_{\phi(x_i)^{-1}} \circ \delta'$ as maps $\mathcal{R}_{\mathbb{C}[G']} \to \mathcal{U}$, where $\tau_{(x_i)^{-1}}$ and $\tau_{\phi(x_i)^{-1}}$ denote the automorphisms of $\mathcal{R}_{\mathbb{C}[G']}$ and $\mathcal{U}$, respectively, given by left conjugation by $(x_i)^{-1}$ and $\phi(x_i)^{-1}$. Using the commutativity of the latter diagram, one can show that $\delta' \circ \tau_{(x_i)^{-1}} \circ \iota = \tau_{\phi(x_i)^{-1}} \circ \delta' \circ \iota$, and the epicity of ι the gives the desired result.

Thus, we have a ring homomorphism δ that, by definition, make the following diagram commute

$$\begin{tikzcd} & \mathbb{C}[G] \arrow[dl, "\iota"'] \arrow[dr, "\phi"] & \\ \mathcal{R}_{\mathbb{C}[G]} \arrow[rr, "\delta"] & & \mathcal{U}. \end{tikzcd}$$

As $\mathcal{R}_{\mathbb{C}[G]}$ is simple, δ must be injective, and the epicity of ι and ϕ together with the commutativity of the diagram shows that δ is epic, and hence surjective by Proposition 4.1.14. In other words, δ is an isomorphism of $\mathbb{C}[G]$-rings.

Being rk_G the unique rank function on $\mathcal{R}_{\mathbb{C}[G]}$, we must have $\mathrm{rk}_G = \delta^{\sharp}(\mathrm{rk}'_\omega)$, and hence as rank functions on $\mathbb{C}[G]$,

$$\mathrm{rk} = \phi^{\sharp}(\mathrm{rk}'_\omega) = \iota^{\sharp}(\delta^{\sharp}(\mathrm{rk}'_\omega)) = \iota^{\sharp}(\mathrm{rk}_G) = \mathrm{rk}_G\,.$$

Thus, $\pi_G^\sharp(\mathrm{rk}_G) = \pi_G^\sharp(\mathrm{rk}) = \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i})$, and since this is valid for every non-principal ultrafilter ω on $\mathbb{N}$, this finishes the proof.

Case 3: G is virtually locally indicable.

Let us fix a non-principal ultrafilter ω on $\mathbb{N}$. As in the previous case, there exists a (finitely generated) finite index normal subgroup N of G which is locally indicable.

Let $F' = F * \mathbb{Z}$ denote the free product of F and $\mathbb{Z}$, which is a free group with one more generator, and let M' (resp. M_i') be the smallest normal subgroup of F' containing M (resp. M_i). Then $G' := F'/M' \cong G * \mathbb{Z}$, $G_i' := F'/M_i' \cong G_i * \mathbb{Z}$ (cf. [Mun00, Theorem 68.7]), and M_i' converges to M' in $\mathrm{MG}(F')$. We claim that G' is virtually locally indicable and ICC (whenever G is non-trivial).

For the ICC property of G' one can consult for instance [Pré13, Example 5.15]. For the virtual local indicability, if N' is the smallest normal subgroup of $G * \mathbb{Z}$ containing N and $\mathbb{Z}$, then $(G * \mathbb{Z})/N' \cong G/N$ (cf. [Mun00, Theorem 68.7]) and hence N' has finite index. Moreover, N' can be shown to be isomorphic to the free product of N and $|G : N|$ copies of $\mathbb{Z}$, and hence locally indicable as a free product of locally indicable groups. Since $G' \cong G * \mathbb{Z}$, this gives the result for G'.

The previous case states that $\pi_{G'}^\sharp(\mathrm{rk}_{G'}) = \lim_\omega \pi_{G_i'}^\sharp(\mathrm{rk}_{G_i'})$, and since we have an embedding $\iota : G \to G'$, the remark at the beginning of the section gives $\iota^\sharp(\mathrm{rk}_{G'}) = \mathrm{rk}_G$. Finally, using the commutativity of

$$\begin{tikzcd} \mathbb{C}[F] \arrow[r, "\pi_G"] \arrow[d, "j"'] & \mathbb{C}[G] \arrow[d, "\iota"] \\ \mathbb{C}[F'] \arrow[r, "\pi_{G'}"'] & \mathbb{C}[G'], \end{tikzcd}$$

where j is the inclusion map, and reasoning as in Lemma 5.2.11, one can show that

$$\pi_G^\sharp(\mathrm{rk}_G) = \pi_G^\sharp \iota^\sharp(\mathrm{rk}_{G'}) = j^\sharp \pi_{G'}^\sharp(\mathrm{rk}_{G'}) = \left(\lim_\omega \pi_{G_i'}^\sharp(\mathrm{rk}_{G_i'})\right)\Big|_{\mathbb{C}[F]} = \lim_\omega \pi_{G_i}^\sharp(\mathrm{rk}_{G_i}).$$

Since this is valid for every non-principal ultrafilter ω on $\mathbb{N}$, this finishes the proof of the case and establishes the result. $\square$

5.3 On the universality of the Hughes-free division ring

Up to now we have seen that for a countable locally indicable group G and for a subfield K of $\mathbb{C}$, the division closure $\mathcal{D}_{K[G]}$ of $K[G]$ inside $\mathcal{U}(G)$ is the Hughes-free division $K[G]$-ring of fractions (Corollary 4.4.3). In addition, we have already mentioned a particular case in which the Hughes-free division ring of fractions is universal provided that the latter one exists (Proposition 3.4.26, and [Sán08, Proposition 6.23]), and further situations for which the universal division ring of fractions exists and it is Hughes-free have been studied in [Jai20B].

Regarding the first fact, one can wonder whether $\mathcal{D}_{K[G]}$ is the universal division $K[G]$-ring of fractions (or equivalently, rk_G is universal in $\mathbb{P}_{\mathrm{div}}(K[G])$) when G is countable locally indicable, and in view of the second fact one can at least hope that this is the case if, for some other reason, we already know about the existence of the universal division $K[G]$-ring of fractions.

With the help of Lemma 5.2.6 we shall prove that for every crossed product $E * G$ (where E is a division ring and G is a locally indicable group) for which there exists a Hughes-free division $E * G$-ring of fractions $\mathcal{D}$, the Sylvester matrix rank function on $E * G$ induced by $\mathrm{rk}_{\mathcal{D}}$ is maximal in $\mathbb{P}_{\mathrm{div}}(E * G)$, thus proving the final assertion of the last paragraph. This result will be proved as a consequence of the following analog of Proposition 5.2.12.

Proposition 5.3.1. *Let $E * G$ be a crossed product of a division ring E and a locally indicable group G. Assume that*

(i) There exists a Hughes-free division $E * G$-ring of fractions $\mathcal{D}$.

(ii) There exists $\mathrm{rk} \in \mathbb{P}_{\mathrm{div}}(E * G)$ such that $\mathrm{rk} \geq \mathrm{rk}_{\mathcal{D}}$ as rank functions on $E * G$.

Then, if $(\mathcal{E}, \phi)$ is the epic division envelope of rk and $\varphi = (\iota, \phi)$ denotes the map $E * G \to \mathcal{D} \times \mathcal{E}$ (where ι is the inclusion map), then the division closure of $\varphi(E * G)$ in $\mathcal{D} \times \mathcal{E}$ is a division ring.

Proof. Since the epic division envelope of an integer-valued rank function and the Hughes-free division $E * G$-ring of fractions are unique up to isomorphisms of division $E * G$-rings (Corollary 3.1.17 and Theorem 3.4.23), $\mathcal{D}$ and $\mathcal{E}$ are uniquely determined by the crossed product and the rank rk. Hence, $\mathcal{S} := \mathcal{D} \times \mathcal{E}$ and $\mathcal{D}_{G,\mathcal{S}}$, the division closure of $\varphi(E * G)$ in $\mathcal{S}$, are uniquely determined by the crossed product and rk. For every subgroup $H \leq G$, set $\mathcal{S}_H = \mathcal{D}_H \times \mathcal{E}_H$, where $\mathcal{D}_H$ and $\mathcal{E}_H$ denote, respectively, the division closures of $\iota(E * H)$ and $\phi(E * H)$ in $\mathcal{D}$ and $\mathcal{E}$. Since, as a product of division rings, $\mathcal{S}_H$ is regular, we have $\mathcal{D}_{H,\mathcal{S}} = \mathcal{D}_{H,\mathcal{S}_H}$ by Lemma 3.3.3.

Considering E fixed, we are going to simultaneously prove the result for every locally indicable group G and for every crossed product $E * G$ for which conditions (i) and (ii) of the statement are satisfied, using induction on the complexity. More precisely, we are going to show that if $E * G$ is any crossed product of E with a locally indicable group G satisfying (i) and (ii), and if $\alpha \in \mathrm{Rat}(E^{\times}G)$ realizes the G-complexity of a non-zero element $a \in \mathcal{D}_{G,\mathcal{S}}$, then a is invertible in $\mathcal{D}_{G,\mathcal{S}}$. Since the morphism of rational $E^{\times}G$-semirings $\Phi_{G,\mathcal{S}} : \mathrm{Rat}(E^{\times}G) \to \mathcal{D}_{G,\mathcal{S}}$ is surjective for every choice of G, crossed product $E * G$ and $\mathcal{S}$ (i.e., of rk) by Proposition 4.3.9, this gives the result.

First observe that for every locally indicable group G and for every crossed product $E * G$ for which conditions (i) and (ii) are satisfied, if $\alpha \in \mathrm{Rat}(E^{\times}G)$ satisfies $\mathrm{Tree}(\alpha) = 1_{\mathcal{T}}$ and realizes the G-complexity of a non-zero element $a \in \mathcal{D}_{G,\mathcal{S}}$, then $\alpha \in E^{\times}G$ by Lemma 4.3.7(ii) and hence $a = \Phi_{G,\mathcal{S}}(\alpha) = \varphi(\alpha) \in \varphi(E^{\times}G)$ is invertible.

Now assume that we have a locally indicable group G, a crossed product $E * G$ for which conditions (i) and (ii) are satisfied and an element $\alpha \in \mathrm{Rat}(E^{\times}G)$ with $\mathrm{Tree}(\alpha) > 1_{\mathcal{T}}$ realizing the G-complexity of a non-zero element $a \in \mathcal{D}_{G,\mathcal{S}}$, and that we have already proved that for every locally indicable group G', for every crossed product $E * G'$ for which conditions (i) and (ii) are satisfied, and for every $\beta \in \mathrm{Rat}(E^\times G')$ with $\mathrm{Tree}(\beta) < \mathrm{Tree}(\alpha)$ realizing the G'-complexity of a non-zero element $a' \in \mathcal{D}_{G',\mathcal{S}'}$, a' is invertible in $\mathcal{D}_{G',\mathcal{S}'}$.

As usual, we can assume that α is primitive (see the proof of Theorem 4.3.14), and hence if H is the image of $\mathrm{source}(\alpha)$ by $E^\times G \to E^\times G/E^\times \cong G$, then H is finitely generated, $\alpha \in \mathrm{Rat}(E^\times H)$ and $a \in \mathcal{D}_{H,\mathcal{S}} = \mathcal{D}_{H,\mathcal{S}_H}$ (see Proposition 4.3.9). In particular, α realizes the H-complexity of a and H, as a subgroup of G, is locally indicable.

If H is trivial, then $E * H = E$ and hence $\varphi(E * H) \cong E$ is division closed and $a \in \mathcal{D}_{H,\mathcal{S}_H} = \varphi(E*H) \cong E$ is invertible. Otherwise, there exists a normal subgroup $N \triangleleft H$ such that H/N is infinite cyclic. Let $x \in E^\times H$ be an element whose image under $E^\times H \to E^\times H/E^\times \cong H \to H/N$ generates H/N, and let τ_x denote the automorphism of $E * N$ induced by left conjugation by x.

By Proposition 3.1.20 (see also Lemma 3.3.4) τ_x extends to isomorphisms $\tau_{\mathcal{D}}$ (given by left conjugation by $\iota(x)$) and $\tau_{\mathcal{E}}$ (given by left conjugation by $\phi(x)$) of $\mathcal{D}_N$ and $\mathcal{E}_N$, respectively, such that $\tau_{\mathcal{D}} \circ \iota = \iota \circ \tau_x$ and $\tau_{\mathcal{E}} \circ \phi = \phi \circ \tau_x$. Moreover, the same proposition says that $\phi : E * N \to \mathcal{E}_N$ extends to a homomorphism $\tilde{\phi} : E * N[t^{\pm 1}; \tau_x] \to \mathcal{E}_N[t^{\pm 1}; \tau_{\mathcal{E}}]$ sending $t \mapsto t$, and that if $j : \mathcal{E}_N[t^{\pm 1}; \tau_{\mathcal{E}}] \to \mathcal{E}_N(t; \tau_{\mathcal{E}})$ is the inclusion map, then $\widetilde{\mathrm{rk}_{|E*N}}$, as a rank function on $E * N[t^{\pm 1}; \tau_x]$, is integer-valued with epic division envelope $(\mathcal{E}_N(t;\tau_{\mathcal{E}}), j \circ \tilde{\phi})$.

If we let ψ denote the $E * N$-isomorphism $\psi : E * H \to E * N[t^{\pm 1}; \tau_x]$ acting as the identity on $E * N$ and sending $x \mapsto t$, then $(\psi^{-1})^\sharp(\mathrm{rk}_{|E*H})$ is a Sylvester matrix rank function on $E * N[t^{\pm 1}; \tau_x]$ that extends $\mathrm{rk}_{|E*N}$, and therefore Lemma 5.2.6 tells us that $\widetilde{\mathrm{rk}_{|E*N}} \geq (\psi^{-1})^\sharp(\mathrm{rk}_{|E*H})$. Thus, since ψ is an isomorphism, the Sylvester matrix rank function $\mathrm{rk}' = \psi^\sharp\left(\widetilde{\mathrm{rk}_{|E*N}}\right)$ on $E * H$ satisfies $\mathrm{rk}' \geq \mathrm{rk}_{|E*H}$. In addition, since the unique Sylvester matrix rank function on $\mathcal{D}_H$ is the restriction of $\mathrm{rk}_{\mathcal{D}}$, and $\mathrm{rk} \geq \mathrm{rk}_{\mathcal{D}}$ as rank functions on $E * G$ by condition (ii), we also have $\mathrm{rk}_{|E*H} \geq \mathrm{rk}_{\mathcal{D}_H}$ as rank function on $E * H$. Adding everything up, we have the following on $E * H$:

- The rank function rk', with epic division envelope $(\mathcal{E}_N(t;\tau_{\mathcal{E}}), j \circ \tilde{\phi} \circ \psi)$. Let us denote $\mathcal{E}' = \mathcal{E}_N(t;\tau_{\mathcal{E}})$ and $\phi' = j \circ \tilde{\phi} \circ \psi$.

- The rank function $\mathrm{rk}_{|E*H}$ with epic division envelope $(\mathcal{E}_H, \phi)$, where we see here ϕ as a map $E * H \to \mathcal{E}_H$.

- The rank function $\mathrm{rk}_{\mathcal{D}_H}$, with epic division envelope $(\mathcal{D}_H, \iota)$, where ι is the inclusion. Moreover, since $\mathcal{D}$ is Hughes-free for $E * G$, $\mathcal{D}_H$ is the Hughes-free division $E * H$-ring of fractions.

- The relation $\mathrm{rk}' \geq \mathrm{rk}_{|E*H} \geq \mathrm{rk}_{\mathcal{D}_H}$ as rank functions on $E * H$.

If we denote $\mathcal{B} = \mathcal{D}_H \times \mathcal{E}'$ and $\mathcal{K} = \mathcal{D}_H \times \mathcal{E}_H \times \mathcal{E}'$, then Corollary 5.2.3 (and its proof) tells us that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Rat}(E^\times H) \arrow[dl, "\Phi_{H,\mathcal{S}_H}"'] \arrow[d, "\Phi_{\mathcal{K}}"] \arrow[dr, "\Phi_{H,\mathcal{B}}"] & \\ \mathcal{D}_{H,\mathcal{S}_H} & \mathcal{D}_{H,\mathcal{K}} \arrow[l, "\pi_{12}"'] \arrow[r, "\pi_{13}"] & \mathcal{D}_{H,\mathcal{B}}, \end{tikzcd}$$

that $a' = \Phi_{H,\mathcal{B}}(\alpha)$ is non-zero (since $a = \Phi_{H,\mathcal{S}_H}(\alpha)$ is non-zero) and that a is invertible if and only if a' is invertible. Here, if $\varphi' = (\iota, \phi')$, then $\mathcal{D}_{H,\mathcal{B}}$ denotes the division closure of $\varphi'(E * H)$ in $\mathcal{B}$.

Observe that, by definition, $\mathrm{Tree}_H(a') \leq \mathrm{Tree}(\alpha)$. Moreover, note that $E * H$ is a crossed product of E with the locally indicable group H and that by the previous observations $E * H$ satisfies conditions (i) and (ii) with respect to $\mathcal{D}_H$ and rk'. Thus, by the induction hypothesis we have that every non-zero $b \in \mathcal{D}_{H,\mathcal{B}}$ with $\mathrm{Tree}_H(b) < \mathrm{Tree}(\alpha)$ is invertible. In particular, if $\mathrm{Tree}_H(a') < \mathrm{Tree}(\alpha)$, we obtain that a', and hence a by the previous reasoning, is invertible.

It is left to study the case in which $\mathrm{Tree}_H(a') = \mathrm{Tree}(\alpha)$, i.e., in which α realizes the H-complexity of a'. For this case, we want to apply as usual Proposition 4.3.13, and for that we need conditions (i), (ii) and (iii) of that proposition to be satisfied.

(i) Consider $\mathcal{A} = \mathcal{S}_N = \mathcal{D}_N \times \mathcal{E}_N$, which is a regular ring, together with the restriction $\varphi : E * N \to \mathcal{A}$.

(ii) By definition of $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{E}}$, $\tau = (\tau_{\mathcal{D}}, \tau_{\mathcal{E}})$ is an automorphism of $\mathcal{A}$ such that $\tau \circ \varphi = \varphi \circ \tau_x$.

(iii) Take $\mathcal{P} = \mathcal{A}((t;\tau))$.

Since $\mathcal{D}$ is Hughes-free, we have an isomorphism of $E * H$-rings $\mathcal{D}_H \to \mathcal{D}_N(t;\tau_{\mathcal{D}})$ (see the proof of Proposition 3.4.31), and this gives a natural embedding $j' : \mathcal{D}_H \to \mathcal{D}_N((t;\tau_{\mathcal{D}}))$. If we denote by j'' the embedding $\mathcal{E}' \to \mathcal{E}_N((t;\tau_{\mathcal{E}}))$, and by Δ the isomorphism $\mathcal{D}_N((t;\tau_{\mathcal{D}})) \times \mathcal{E}_N((t;\tau_{\mathcal{E}})) \cong \mathcal{A}((t;\tau))$, one can carefully show that the following diagram commutes

$$\begin{tikzcd}
E * N \arrow[r] \arrow[d, "\varphi"'] & E * H \arrow[r, "\varphi'"] \arrow[d, "\tilde{\varphi}"] & \mathcal{B} \arrow[d, "\Delta\circ(j',j'')"] \\
\mathcal{A} \arrow[r] & \mathcal{A}((t;\tau)) \arrow[r, "\mathrm{id}"'] & \mathcal{P},
\end{tikzcd}$$

where $\tilde{\varphi}$ is the map acting as φ on $E * N$ and sending $x \mapsto t$.

As $\mathcal{B}$ is regular and $\Delta \circ (j', j'')$ is injective, we have by Lemma 3.3.3 that $\mathcal{D}_{H,\mathcal{B}} \cong \mathcal{D}_{H,\Delta\circ(j',j'')(\mathcal{B})} = \mathcal{D}_{H,\mathcal{P}}$ as $E*H$-rings, where $\mathcal{D}_{H,\mathcal{P}}$ denotes the division closure of $\tilde{\varphi}(E*H)$ in $\mathcal{P}$. Consequently, Lemma 4.3.11 tells us that

$$\begin{tikzcd}
 & \mathrm{Rat}(E^{\times} H) \arrow[dl, "\Phi_{H,\mathcal{B}}"'] \arrow[dr, "\Phi_{H,\mathcal{P}}"] & \\
\mathcal{D}_{H,\mathcal{B}} \arrow[rr, "\cong"'] & & \mathcal{D}_{H,\mathcal{P}}
\end{tikzcd}$$

commutes and that the H-complexity of an element in $\mathcal{D}_{H,\mathcal{B}}$ coincides with the H-complexity of its image in $\mathcal{D}_{H,\mathcal{P}}$ because it is realized by the same element of $\mathrm{Rat}(E^{\times} H)$. In particular, if a'' is the image of a' in $\mathcal{D}_{H,\mathcal{P}}$, then α realizes its H-complexity, and since every non-zero $b \in \mathcal{D}_{H,\mathcal{B}}$ with $\mathrm{Tree}_H(b) < \mathrm{Tree}(\alpha) = \mathrm{Tree}_H(a')$ was invertible by the induction hypothesis, the same holds for any non-zero element in $\mathcal{D}_{H,\mathcal{P}}$ strictly less H-complex than a''.

Thus, Proposition 4.3.13 applies and says that $a'' = \sum_k a_k \in \mathcal{D}_{N,\mathcal{A}}((t;\tau))$ with $a_k \in \mathcal{D}_{N,\mathcal{A}} t^k$ and $\mathrm{Tree}_H(a_k) \leq \mathrm{Tree}_H(a'')$. Moreover, we claim that there are at least two non-zero summands. Otherwise, if $a'' = a_n$, then $\mathrm{Tree}_H(a'') = \mathrm{Tree}_H(a_n)$, from where the same proposition tells us that $\alpha \in \mathrm{Rat}(E^\times N)x^n \subseteq \mathrm{Rat}(E^\times N)E^\times G$. This would imply by Theorem 4.3.8(iv) that $\mathrm{source}(\alpha) \leq E^\times N$, and hence that $H \leq N$, a contradiction.

Hence, $\mathrm{Tree}_H(a_k) < \mathrm{Tree}_H(a'')$ for all k and therefore each non-zero a_k is invertible in $\mathcal{D}_{H,\mathcal{P}}$ by the discussion above. In particular this is true for n, the smallest k such that a_k is non-zero. Therefore $a_n t^{-n}$ is invertible in $\mathcal{D}_{N,\mathcal{A}}$ and consequently a'' is invertible in $\mathcal{P}$, and hence in $\mathcal{D}_{H,\mathcal{P}}$ since it is division closed. Therefore, a' is invertible in $\mathcal{D}_{H,\mathcal{B}}$, and this implies that a is invertible in $\mathcal{D}_{H,\mathcal{S}_H} \subseteq \mathcal{D}_{G,\mathcal{S}}$, as we wanted to show. □

With this proposition, the proof of the maximality of $\mathrm{rk}_{\mathcal{D}}$ goes as follows.

Theorem 5.3.2. *Let $E * G$ be a crossed product of a division ring E and a locally indicable group G, and assume that there exists a Hughes-free division $E * G$-ring of fractions $\mathcal{D}$. Then the Sylvester matrix rank function $\mathrm{rk}_{\mathcal{D}}$ is maximal in $\mathbb{P}_{\mathrm{div}}(E * G)$.*

Proof. Assume that $\mathrm{rk} \in \mathbb{P}_{\mathrm{div}}(E*G)$ is a rank function such that $\mathrm{rk} \geq \mathrm{rk}_{\mathcal{D}}$, and let $(\mathcal{E}, \phi)$ denote its epic division envelope. If we set $\mathcal{S} = \mathcal{D} \times \mathcal{E}$ and $\varphi = (\iota, \phi) : E * G \to \mathcal{S}$, then Proposition 5.3.1 tells us that the division closure $\mathcal{D}_{G,\mathcal{S}}$ of $\varphi(E * G)$ in $\mathcal{S}$ is a division ring. If $\pi_1 : \mathcal{D}_{G,\mathcal{S}} \to \mathcal{D}$ and $\pi_2 : \mathcal{D}_{G,\mathcal{S}} \to \mathcal{E}$ denote the restrictions of the canonical projections from $\mathcal{S}$ to each factor, then since $\mathcal{D}_{G,\mathcal{S}}$ is a division ring, we have that they are both injective. Moreover, as the following diagrams commute

$$
\begin{tikzcd}
E * G \arrow[r, "\varphi"] \arrow[d, "\iota"'] & \mathcal{D}_{G,\mathcal{S}} \arrow[d, "\pi_1"] \\
\mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
E * G \arrow[r, "\varphi"] \arrow[d, "\phi"'] & \mathcal{D}_{G,\mathcal{S}} \arrow[d, "\pi_2"] \\
\mathcal{E} \arrow[r, "\mathrm{id}"'] & \mathcal{E}
\end{tikzcd}
$$

with φ, ι and ϕ epic, and $\mathcal{D}_{G,\mathcal{S}}, \mathcal{D}$ and $\mathcal{E}$ division rings (in particular, regular), Corollary 4.1.15 tells us that $\pi_1(\mathcal{D}_{G,\mathcal{S}}) = \mathcal{D}$ and $\pi_2(\mathcal{D}_{G,\mathcal{S}}) = \mathcal{E}$. Therefore, π_1 and π_2 are also surjective, and hence isomorphisms.

From here, $\delta = \pi_2 \circ \pi_1^{-1}$ defines an isomorphism from $\mathcal{D}$ to $\mathcal{E}$ making the following commute

$$
\begin{tikzcd}
 & E * G \arrow[dl, "\iota"'] \arrow[dr, "\phi"] & \\
\mathcal{D} \arrow[rr, "\delta"] & & \mathcal{E}
\end{tikzcd}
$$

Since in a division ring there exists only one rank function, we have that $\mathrm{rk}_{\mathcal{D}} = \delta^\sharp(\mathrm{rk}_{\mathcal{E}})$, and consequently as rank functions on $E * G$,

$$\mathrm{rk} = \phi^\sharp(\mathrm{rk}_{\mathcal{E}}) = \iota^\sharp(\delta^\sharp(\mathrm{rk}_{\mathcal{E}})) = \iota^\sharp(\mathrm{rk}_{\mathcal{D}}) = \mathrm{rk}_{\mathcal{D}},$$

what finishes the proof. □

Corollary 5.3.3. *Let* $E * G$ *be a crossed product of a division ring* E *and a locally indicable group* G*. If there exist a Hughes-free division* $E * G$*-ring of fractions and a universal* $E * G$*-ring of fractions, then they are isomorphic as* $E * G$*-rings. In particular, if* G *is countable,* K *is a subfield of* $\mathbb{C}$ *and there exists a universal* $K[G]$*-ring of fractions, then it is isomorphic to* $\mathcal{D}_{K[G]}$ *as a* $K[G]$*-ring.*

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